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RESEARCH ARTICLE

Equitable Common Neighbour Equitable Domination in Graphs

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ABSTRACT

The concept of equitability can be traced to the paper titled "Equitable coloring" by W.Meyer. In that paper the cardinalities of the color class should differ by at most one. Later Prof. E. Sampath Kumar introduced the idea of degree equitability. Two vertices are degree equitable if their degrees differ by at most one. Based on this idea several papers were published. Equitable domination has been defined and studied. Anwar Alwardi et al. gave importance to common neighbourhood of pairs of vertices. Several papers were published by them on common neighbourhood domination, injective domination and injective equitable domination. In this paper equitable common neighbour of pairs of vertices are considered and equitable common neighbour equitable domination is introduced and studied.

Keywords: Domination, equitable domination, common neighborhood domination,
2010 Mathematics subject classification: 05C69.

INTRODUCTION

Graph theoretical terminologies not given here can be founded in [5, 7, 9]. Let $G = (V, E)$ be a simple graph. The neighbourhood of a vertex v, denoted by $N(v)$, is the set of all vertices adjacent to v in G. If v is a vertex of G then the integer $deg(v) = |N(v)|$ is said to be the degree of v in G. The minimum and maximum degree among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex of degree one in a graph is called a pendent vertex or an end vertex. A support is the unique neighbour of an end-vertex.

A set $D \subseteq V(G)$ is a dominating set in G if for every vertex $v \in V(G) - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$.The domination number of a graph G, denoted $\gamma(G)$, is the cardinality of a minimum dominating set of G. The concept of equitability was originally conceived in proper colouring of vertices where the cardinalities of any two colour classes differ by at most one [12]. E. Sampathkumar initiated the concept of degree equitability in the vertex set of a graph. Two vertices are said to be degree equitable if their degrees differ by at most one. A subset D of $V(G)$ is called an equitable dominating set of G if for any v in $V-D$, there exists u in D such that u and v are adjacent and degree equitable [2, 6, 11]. A subset D of V is called a common neighbourhood dominating set if for every v in $V-D$, there exists a vertex u in D such that u and v are adjacent and u and v have atleast one common neighbour. The minimum cardinality of such a dominating set is called common neighbourhood domination number of G and is denoted by γ_{cn} [3].

Definition 1.1 : Let G be a simple graph. A subset $S \subseteq V(G)$ is called an equitable common neighbour equitable dominating set (ecne-dominating set) if for every vertex $u \in V-S$ there exists $v \in S$ such that u and v are equitable (not necessarily adjacent) and have an equitable common neighbour. The minimum (maximum) cardinality of an equitable common neighbour equitable dominating set is called an equitable common neighbour equitable domination number of G and is denoted by $\gamma_e^{ecn}(G)(\Gamma_e^{ecn}(G))$.

Remark 1.2: The property of an equitable common neighbour equitable domination is super hereditary.

γ_e^{ecn} of some standard graphs

1. $\gamma_e^{ecn}(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1 & if n \equiv 2,3 (mod\ 6) \\ \left\lceil \frac{n}{3} \right\rceil & if n \not\equiv 2,3 (mod\ 6) \end{cases}$
2. $\gamma_e^{ecn}(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1 & if n \equiv 2 (mod\ 6) \\ \left\lceil \frac{n}{3} \right\rceil & if n \not\equiv 2 (mod\ 6) \end{cases}$
3. $\gamma_e^{ecn}(K_n) = 1$
4. $\gamma_e^{ecn}(K_{m,n}) = \begin{cases} 2 & if\ |m-n| \le 1 \\ m+n & if\ |m-n| \ge 2 \end{cases}$
5. $\gamma_e^{ecn}(K_{1,n}) = n+1, n \ge 3$
6. $\gamma_e^{ecn}(W_n) = \begin{cases} 1 & if n = 4 \\ n & if n \ge 5 \end{cases}$
7. $\gamma_e^{ecn}(D_{r,s}) = \begin{cases} 1 & if r = 1, s = 0 \\ 2 & if r = 1, s = 1 \\ 4 & if r = 2, s = 0,1 \\ r+s+2 & if r, s \ge 2 \end{cases}$
8. $\gamma_e^{ecn}(K_{a_1,a_2,\cdots,a_n}) = 2r$, where r is the number of equitable partition of $a_1, a_2, \cdots, a_n$
9. $\gamma_e^{ecn}(K_m(a_1, a_2, \cdots, a_n) = \sum_{i=1}^{m} a_i + 1$

Definition 1.3 : A subset $S \subseteq V(G)$ is called an ecne-independent set if no two vertices in S are equitable and have an equitable common neighbour.Clearly, this property is hereditary.

Remark 1.4 : Any maximal ecne-independent set is a minimal ecne-dominating set.

Results on $\gamma_e^{ecn}(G)$

Definition 2.1: Let $S \subseteq V(G)$. The ecn private equitable neighbour of $u \in S$ denoted by$pn_e^{ecn}(S,u)$ is defined as

$$pn_e^{ecn}(S,u) = \begin{Bmatrix} v \in V: \text{v and u have a common equitable neighbour and for any,} \\ w \in S-\{u\}, \text{v and w do not have an equitable common neighbour} \end{Bmatrix}$$

Proposition 2.2: Let $S \subseteq V$ be an ecne-dominating set of G. S is a minimal ecne-dominating set if and only if for any $u \in S$, $pn_e^{ecn}(S, u) \neq \varphi$.

Definition 2.3: Let $u \in V(G)$. The ecne-neighbourhood of u denoted by $N_e^{ecn}(u)$ is defined as $N_e^{ecn}(u) = \{v \in V$: uandvareequitableandhaveanequitablecommonneighbour$\}$. Theecne-degree of a vertex is defined as $deg_e^{ecn}(u) = |N_e^{ecn}(u)|$. The maximum, minimumecne-degree of G are denoted by $\Delta_e^{ecn}(G), \delta_e^{ecn}(G)$ and defined as $\Delta_e^{ecn}(G) = \max\{deg_e^{ecn}(u): u \in V\}$ and $\delta_e^{ecn}(G) = \min\{deg_e^{ecn}(u): u \in V\}$.

Definition 2.4: A vertex $u \in V$ is called an ecne-isolate of G if u and v have no common equitable neighbour in G for every $v \in V(G) - \{u\}$.

Remark 2.5: Every equitable isolate is an ecne-isolate but not the converse.

Remark 2.6: Any ecne-dominating set contains all ecne-isolates of G.

Remark 2.7: If u is an ecne-isolate, then $\delta_e^{ecn}(G) = 0$

Remark 2.8: If $\delta_e^{ecn}(G) \geq 1$, then G has no ecne-isolates and hence $\gamma_e^{ecn}(G) \leq \frac{n}{2}$

Proposition 2.9: For any simple graph G with no ecne-isolates, $\left\lceil \frac{n}{1+\Delta_e^{ecn}(G)} \right\rceil \leq \gamma_e^{ecn}(G) \leq n - \Delta_e^{ecn}(G)$.

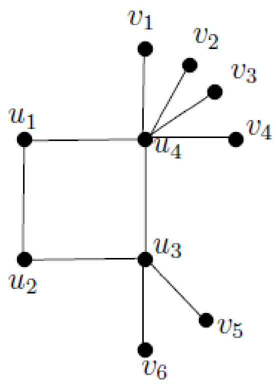
Example 2.10: When $G \simeq P_n$, $n \equiv 0,1 (mod 3)$, $\Delta_e^{ecn}(P_n) = 2$ and hence $\left\lceil \frac{n}{1+\Delta_e^{ecn}(G)} \right\rceil = \left\lceil \frac{n}{3} \right\rceil = \gamma_e^{ecn}(P_n)$.
When $G \simeq C_4$ and $G \simeq C_5$, $\Delta_{ecn}(C_4) = \Delta_e^{ecn}(C_5) = 2$ and hence $\gamma_e^{ecn}(C_4) = 2 = n - \Delta_e^{ecn}(G)$ and $\gamma_e^{ecn}(C_5) = 3 = n - \Delta_e^{ecn}(G)$. Thus, the equality holds in Proposition 2.9.

Proposition 2.11: For any graph G with n vertices, $\gamma_e^{ecn}(G) = n$ if and only if every vertex in G is either an equitable isolate or an ecne-isolate of G.

Example 2.12: The above result holds for, $K_{1,n}$, $n \geq 3$

Remark 2.13: There exists a graph G in which a vertex u is not an equitable isolate but u is an ecne-isolate of G.

Example 2.14



G

Here $\gamma_e^{ecn}(G) = 10 = |V(G)|$. In the above graph G, u_1 and u_2 are not equitable isolates but both are ecne-isolates.

Theorem 2.15: A graph G has a unique minimal ecne-dominating set if and only if the set of all ecne-isolates forms an ecne-dominating set.

Proof. Suppose G has a unique minimal ecne-dominating set, say D. Let S be the set of all ecne-isolates of G. Then $S \subseteq D$. Suppose there exists $u \in D - S$. Then u is not an ecne-isolate. Therefore, $V - \{u\}$ is an ecne-dominating set. Thus, there exists a minimal dominating set $D_1 \subseteq V - \{u\}$, which contradicts that D is unique. Hence $S = D$. Conversely, if the set D of ecne-isolates of G form an ecne-dominating set, then any ecne-dominating set of G contains D. Therefore, G has a unique minimal ecne-dominating set.

Theorem 2.16: For any (n,m) - graph G, $\gamma_e^{ecn}(G) \geq n - m$
Proof. Let D be a γ_e^{ecn}-set of G. Since every vertex in $V - D$ has equitable common neighbour with some vertex of D, $m \geq |V - D|$. Therefore, $|D| \geq n - m$.

Theorem 2.17: Let G be a graph without ecne-isolates. Then, the complement of a minimalecne-dominating set is also an ecne-dominating set.

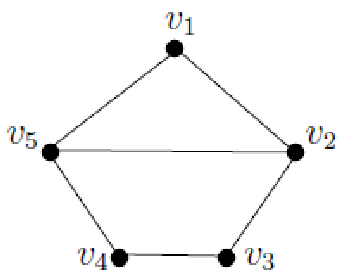
Corollary 2.18: If G has no ecne-isolates, then $\gamma_e^{ecn}(G) \leq \frac{n}{2}$

Definition 2.19: Let $u \in V(G)$. $e_{eq}(u)$ is the maximum set of vertices $u_1, u_2, \cdots, u_k$ such that $u_1 u_2 \cdots u_k$ is a path such that u_i and u_{i+2} have u_{i+1} as an equitable common neighbour in G. The maximum (minimum) value of $e_{eq}(u)$ is called equitable diameter (equitable radius) of G and denoted by $diam_{eq}(G)(r_{eq}(G))$.

Theorem 2.20: Let G be a simple graph. $\gamma_e^{ecn}(G) = 1$ if and only if there exists a vertex$u \in V(G)$ such that u is equitable with every other vertex of G and $N(u) \subseteq N_e^{ecn}(u)$ and $e_{eq}(u) \leq 2$.

Proof. Suppose $\gamma_e^{ecn}(G) = 1$. Then there exists $u \in V(G)$ such that u is equitable with every other vertex of G and for any u and v, have a common equitable neighbour. Suppose u and v are adjacent. Then they have an equitable common neighbour and so $N(u) \subseteq N_e^{ecn}(u)$. Suppose u and v are not adjacent, let w be the equitable common neighbour of u and v. Then, $e_{eq}(u) \leq 2$. Conversely, suppose u satisfies the hypothesis. Then for any $v \in V(G)$, u and v are equitable. Since $e_{eq} \leq 2$, either u and v are adjacent or u and v are at a distance 2. Since $N(u) \subseteq N_e^{ecn}(u)$ any adjacent vertex of u has a equitable common neighbour with u. Suppose u and v are not adjacent. Then $e_{eq}(u) = 2$ and there exists $w \in V(G)$ such that u and v have w as an equitable common neighbour. Therefore, $\gamma_e^{ecn}(G) = 1$.

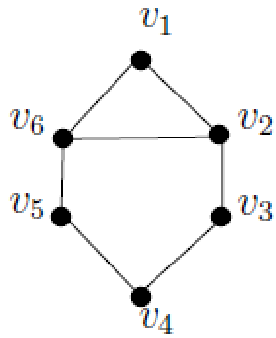
Example 2.21



G

For the above graph G, $\gamma_e^{ecn}(G) = 1$, and $\{v_1\}$ is an ecne-dominating set. Clearly v_1 is equitable with v_2,v_3,v_4 and v_5. Since v_2 and v_5 have equitable common neighbour with v_1, $\{v_2, v_5\} \in N_e^{ecn}(1)$. Therefore, $N(1) \subseteq N_e^{ecn}(1)$. Also, $e_{eq}(v_1) \leq 2$, since v_3 and v_4 are at distance 2 from v_1 and $\{v_1, v_2, v_3\}$ and $\{v_1, v_5, v_4\}$ are ecne-paths.

Example 2.22



H

For the above graph H, $\gamma_e^{ecn}(H) \geq 2$. Also the vertex v_1 is equitable with all other vertices, $N(1) \subseteq N_e^{ecn}(1)$ but $e_{eq}(v_1) \geq 3$, since $v_1v_2v_3v_4$ is an ecne-path. Note that if v_1 and v_3 or v_1 and v_5 are made adjacent in the above graph H, then $\gamma_e^{ecn}(H) = 1$ and $\{v_1\}$ is an ecne-dominating set.

Theorem 2.23: If $\text{diam}_{eq}(G) \leq 3$, then $\gamma_e^{ecn}(G) \leq \Delta_e^{ecn}(G) + 1$.

Proof. If $diam_{eq}(G) = 1$, then $G \simeq K_n$ and $\Delta_e^{ecn}(G) = n - 1$. Hence, $\gamma_e^{ecn}(G) = 1 < n = \Delta_e^{ecn}(G) + 1$. Suppose, $diam_{eq}(G) = 2$ or 3. Then the set of vertices v, $(deg_{ecn}(v) = \Delta_e^{ecn}(G))$ and which have equitable distance 1, 2 or 3 from v is an ecne-dominating set of G of order $\Delta_e^{ecn}(G) + 1$. Therefore, $\gamma_e^{ecn}(G) \leq \Delta_e^{ecn}(G) + 1$.

Theorem 2.24: For any graph G such that G and $\overline{G}$ have no ecne-isolates, $\gamma_e^{ecn}(G) + \gamma_e^{ecn}(\overline{G}) \leq n$.

Proof. Suppose G has an ecne-isolated vertex. Then $\gamma_e^{ecn}(\overline{G}) = 1$ and $\gamma_e^{ecn}(G) \leq n$. Therefore, $\gamma_e^{ecn}(G) + \gamma_e^{ecn}(\overline{G}) \leq n + 1$. A similar result holds when $\overline{G}$ has an ecne-isolate. Let G and $\overline{G}$ have no ecne-isolated vertex. Then $\gamma_e^{ecn}(G) \leq \frac{n}{2}$ and $\gamma_e^{ecn}(\overline{G}) \leq \frac{n}{2}$. Therefore, $\gamma_e^{ecn}(G) + \gamma_e^{ecn}(\overline{G}) \leq n$.

Theorem 2.25: If G and $\overline{G}$ have no ecne-isolates and if $\gamma_e^{ecn}(G) \cdot \gamma_e^{ecn}(\overline{G}) \leq n$, then $\gamma_e^{ecn}(G) + \gamma_e^{ecn}(\overline{G}) \leq \left\lfloor \frac{n}{2} \right\rfloor + 2$.

Proof. Since G and $\overline{G}$ have no ecne-isolates, $\gamma_e^{ecn}(G) \leq \frac{n}{2}$&$\gamma_e^{ecn}(\overline{G}) \leq \frac{n}{2}$. Clearly,$\gamma_e^{ecn}(G) \geq 2$&$\gamma_e^{ecn}(\overline{G}) \geq 2$. If $\gamma_e^{ecn}(G) = 2$ or $\gamma_e^{ecn}(\overline{G}) = 2$, then $\gamma_e^{ecn}(G) + \gamma_e^{ecn}(\overline{G}) \leq \frac{n}{2} + 2$. Suppose $\gamma_e^{ecn}(G) \geq 4$&$\gamma_e^{ecn}(\overline{G}) \geq 4$. Since $\gamma_e^{ecn}(G) \cdot \gamma_e^{ecn}(\overline{G}) \leq n$, $\gamma_e^{ecn} \leq \left\lfloor \frac{n}{\gamma_e^{ecn}(\overline{G})} \right\rfloor$ and $\gamma_e^{ecn}(\overline{G}) \leq \left\lfloor \frac{n}{\gamma_e^{ecn}} \right\rfloor$. Therefore, $\gamma_e^{ecn}(G) + \gamma_e^{ecn}(\overline{G}) \leq \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor \leq 2\left\lfloor \frac{n}{4} \right\rfloor < \frac{n}{2} + 2$. Suppose $\gamma_e^{ecn}(G) = 3$ or $\gamma_e^{ecn}(\overline{G}) = 3$. Then, $\gamma_e^{ecn}(\overline{G}) \leq \left\lfloor \frac{n}{3} \right\rfloor$, that is $\gamma_e^{ecn}(G) + \gamma_e^{ecn}(\overline{G}) \leq 3 + \left\lfloor \frac{n}{3} \right\rfloor$. Since $\gamma_e^{ecn}(G) = 3 \leq \left\lfloor \frac{n}{2} \right\rfloor$, $n \geq 6$. Therefore, $\gamma_e^{ecn}(G) + \gamma_e^{ecn}(\overline{G}) \leq 3 + \left\lfloor \frac{n}{3} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor + 2$.

Equitable common neighbourhood dominating parameters

Definition 3.1: Let $S \subseteq V(G)$. If $\text{epn}_{ecn}(S, u) \neq \varphi$, for any $u \in S$, then S is called an ecne-irredundant set of G.

Remark 3.2: The property of ecne-irredundance is hereditary.

Theorem 3.3: Any minimal ecne-dominating set is a maximal ecne-irredundant set.

Proof. Let S be a minimal ecne-dominating set. Then S is an ecne-irredundant set of G. Suppose for any $u \in V - S$, $S \cup \{u\}$ is an ecne-irredundant set. Then, $pn_{ecn}(S \cup \{u\}, u) \neq \phi$. Since S is an ecne-dominating set, there exists $v \in S$ such that u and v have a common equitable private neighbour. Since $pn_{ecn}(S \cup \{u\}, u) \neq \phi$, there exists no $w \in (S \cup \{u\}) - \{u\} = S$ such that u and w have equitable common neighbour, which is a contradiction. Thus, S is a maximal ecne-irredundant set of G.

Definition 3.4: The minimum (maximum) cardinality of a maximal ecne-irredundant set of G is denoted by $ir_e^{ecn}(G)(IR_e^{ecn}(G))$.

Remark 3.5: $ir_e^{ecn}(G) \leq \gamma_e^{ecn}(G) \leq \Gamma_e^{ecn}(G) \leq IR_e^{ecn}(G)$

Proposition 3.6: For any graph G, $\frac{\gamma_e^{ecn}(G)}{2} < ir_e^{ecn}(G) \leq \gamma_e^{ecn}(G) \leq 2ir_e^{ecn}(G) - 1$

Definition 3.7: The minimum cardinality of an independent ecne-dominating set of G is denoted by $i_e^{ecn}(G)$.

Remark 3.8: $\gamma_e^{ecn}(G) \leq i_e^{ecn}(G)$

Definition 3.9: The maximum cardinality of an independent ecne-set of G is denoted by $\beta_e^{ecn}(G)$.

Remark 3.10: $ir_e^{ecn}(G) \leq \gamma_e^{ecn}(G) \leq i_e^{ecn}(G) \leq \beta_e^{ecn}(G) \leq \Gamma_e^{ecn}(G) \leq IR_e^{ecn}(G)$

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