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EDGE-CRITICALITY IN UNICYCLIC GRAPHS UPON OUTER CONNECTED INDEPENDENCE

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Abstract

In a graph $G = (V, E)$ (not necessarily be connected) an independent set $S \subseteq V$ is said to be an outer connected independent set if $\omega(G - S) \leq \omega(G)$, where $\omega(G)$ is the number of components in G. The maximum cardinality of an outer connected independent set is called the outer connected independence number and it is denoted by $Ioc(G)$. This concept was introduced in [5] and as the continuation of this work, the effect of Ioc upon edge removal has been reported in [6]. This paper is a further study of the critical edges upon Ioc in Unicyclic graphs.

1. Introduction

We consider finite, undirected graphs with neither loops nor multiple edges. For the terms not defined here, refer to the book by Chartrand and Lesniak [1]. For $S \subseteq V(G)$, by $\langle S \rangle$, we mean the subgraph of G induced by S, which is defined to be the subgraph of G with vertex set S where u and v are adjacent if and only if they are adjacent in G. A vertex of degree 1 is called an

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end-vertex or a leaf and an edge incident with an end-vertex is called a pendant edge. Further, the subgraph $\langle V - S \rangle$ of G induced by $V - S$ is written as $G - S$. Also, for an edge $e \in E(G), G - e$ denotes the graph formed from G by removing the edge e from G. The number of components in a graph G is denoted by $\omega(G)$.

A set S of vertices of a graph G is called an independent set of G if no two vertices in S are adjacent in G. The maximum cardinality of an independent set is the independence number $\beta_0(G)$ of G. An outer connected independent set (oci-set) of a graph G is an independent set S of G such that $\omega(G - S) \leq \omega(G)$ and the outer connected independence number denoted by $Ioc(G)$, is the maximum cardinality of an oci-set of G. This concept was introduced in [5]. The study of the effect of removal of an edge from a graph G upon the outer connected independence number $Ioc(G)$, was initiated in [6], where it has been proved that in a connected graph G, $Ioc(G) - 1 \leq Ioc(G - e)$ $\leq Ioc(G) + 2$ for any edge e of G. In this connection, the edge set $E(G)$ of a connected graph has been split into four sets $E^-(G)$, $E^0(G)$, $E_1^+(G)$ and $E_2^+(G)$ as follows.

$$E^-(G) = \{e \in E(G) : Ioc(G - e) = Ioc(G) - 1\}$$

$$E^0(G) = \{e \in E(G) : Ioc(G - e) = Ioc(G)\}$$

$$E_1^+(G) = \{e \in E(G) : Ioc(G - e) = Ioc(G) + 1\}$$

$$E_2^+(G) = \{e \in E(G) : Ioc(G - e) = Ioc(G) + 2\}$$

This paper investigates the properties of these sets in unicyclic graphs. We state the following results that are needed in the subsequent sections.

Theorem 1.1 [5]**.** *Let G be a graph all of whose components are of order at least* 3. *Then* $Ioc(G) \geq l$ *where* l *is the number of pendant vertices of G. Further, equality holds if and only if every vertex of G with degree at least two is a cut-vertex of G.*

Theorem 1.2 [5]**.** *If G is a unicyclic graph then*

$$i_{oc}(G) = I_{oc}(G) = \begin{cases} l & \text{if } r = k \\ l+1 & \text{if } r \neq k \end{cases}$$

Theorem 1.3 [6]. *If H is connected, then $Ioc(H \cup K_1)$ is either $Ioc(H)+1$ or $Ioc(H)+2$.*

Theorem 1.4 [6]. *Let G be a graph with no isolates. If $G_1, G_2, G_3, \ldots, G_k$ are the components of G, then $Ioc(G) = \sum_{i=1}^{k} I_{oc}(G_i)$.*

Theorem 1.5 [6]. *Let G be a connected graph. If $e \in E(G)$ is a bridge, then $Ioc(G) \leq Ioc(G-e) \leq Ioc(G)+2$.*

Theorem 1.6 [7]. *If T is a tree of order at least 3 with l pendant vertices, then $Ioc(T \cup K_1)$ is either $l+1$ or $l+2$ Further $Ioc(T \cup K_1) = l+1$ if and only if every vertex of degree 2 is a support in T.*

Theorem 1.7 [7]. *Let T be a tree with at least 3 vertices and let $e = uv$ be a non-pendant edge of T. Then*

(i) *$e \in E^0(T)$ if and only if $x \in \{u, v\}$ is a support when deg $x = 2$.*

(ii) *$e \in E_1^+(T)$ if and only if exactly one of u and v is non-support with degree 2.*

(iii) *$e \in E_2^+(T)$ if and only if both u and v are non-support vertices with degree 2.*

2. Critical edges

As seen in [6], the edge set of a graph G can be partitioned into sets $E^-(G)$, $E^0(G)$, $E_1^+(G)$ and $E_2^+(G)$ with respect to the outer connected independence. This section completely determines these sets for the classes of all connected unicyclic graphs. Throughout this paper, for a unicyclic graph G with the cycle C, we use l to denote the number of pendant vertices of G, r, k for the length of the cycle C and r for the number of vertices of G with degree greater than 2 that lie on C. We start with the following lemma.

Lemma 2.1. *Let G be a unicyclic graph of order at least* 4 *with l pendant vertices.*

(i) *When* $r = k$, $Ioc(G \cup K_1)$ *is either* $l+1$ *or* $l+2$. *Further* $Ioc(G \cup K_1) = l+1$ *if and only if every vertex of G with degree 2 is a support and every vertex of degree* 3 *lying on C is a support.*

(ii) *When* $r < k$, $Ioc(G \cup K_1)$ *is either* $l+2$ *or* $l+3$. *Further* $Ioc(G \cup K_1) = l+2$ *if and only if every vertex of G with degree* 2 *lying outside C is a support and either there is exactly one vertex of degree* 2 *on C or there are exactly two adjacent vertices of degree* 2 *on C.*

Proof. By Theorem 1.2, we have $Ioc(G) = l$ or $l+1$ according as $r = k$ or $r < k$. Therefore, by Theorem 1.3, when $r = k$, the value of $Ioc(G \cup K_1)$ is either $l+1$ or $l+2$ and when $r < k$, it is either $l+2$ or $l+3$.

(i) Let $r = k$ and let $Ioc(G \cup K_1) = l+1$. Let u be either a vertex of degree 2 or a vertex of degree 3 lying on C. Consider the tree $G - e$, where e is an edge of G that is chosen to be an edge incident with u when deg $u = 3$ or an arbitrary edge otherwise. Now, Theorem 1.6 claims that u should be a support in G.

Conversely, suppose every vertex of degree 2 is support and also every vertex of degree 3 on C is support. Let S be a maximal oci-set of $G \cup K_1$. Certainly, S must contain the isolate vertex of $G \cup K_1$. Also S cannot have two vertices x and y such that $\{\deg x, \deg y\} \subseteq \{2, 3\}$. Further, if S contains a vertex x which is either a vertex of degree 2 or a vertex on C with degree 3, then all the pendant vertices of T other than the only pendant neighbor of x must be in S. Thus $|S| = l+1$. That is, $Ioc(G \cup K_1) = l+1$. If S contains neither a vertex of degree 2 nor a vertex on C with degree 3, then all the pendant vertices of G must be in S. Hence $Ioc(G \cup K_1) = l+1$.

(ii) Let $r < k$ and $Ioc(G \cup K_1) = l+2$. If there exists a non-support vertex u not on C with $\deg u = 2$, then the set consisting of all the pendant vertices of G together with the isolate vertex of $G \cup K_1$, the vertex u and a vertex on C with degree 2 forms an oci-set of $G \cup K_1$ with cardinality $l+3$, which is a contradiction. Further, if C contains two non-pendant vertices of G and the isolate vertex of $G \cup K_1$, form an oci-set of $G \cup K_1$ with cardinality $l + 3$, again a contradiction. Thus, every vertex of G with degree 2 lying outside C is a support and either there is exactly one vertex of degree 2 on C or there are exactly two adjacent vertices of degree 2 on C.

Conversely, suppose every vertex of degree 2 outside C is a support and there is either exactly one vertex of degree 2 on C or exactly two adjacent vertices of degree 2 on C. Let S be a maximal oci-set of $G \cup K_1$. Certainly, S must contain the isolate vertex of $G \cup K_1$ and a vertex of degree 2 lying on C. Further S can have at most one vertex of G with degree 2 lying outside C. If S contains a vertex x of degree 2 lying outside C then the only pendant neighbor of x cannot be in S; whereas all the remaining pendant vertices of T must be in S. Thus $|S| = l + 2$. That is $Ioc(G \cup K_1) = l + 2$. Otherwise all the pendant vertices of G are in S and so $Ioc(G \cup K_1) = l + 2$.

Let us now proceed to investigate the properties of edges belonging to each of the sets $E^-(G)$, $E^0(G)$, $E_1^+(G)$ and $E_2^+(G)$.

Theorem 2.2. *Let G be a unicyclic graph with order $n \geq 5$ with $r < k$. Let $e = uv$ be a pendant edge of G with $\deg v = 1$. Then $e \in E^0(G) \cup E_1^+(G) \cup E_2^+(G)$. Further,*

(a) *$e \in E^0(G)$ if and only if the following conditions hold.*

(i) *$\deg u \geq 3$ and when deg $u = 3$ outside C, the vertex u has a pendant neighbor other than v and when deg $u = 3$ on C, one of the neighbors of u other than v is the only possible vertex of degree 2 on C.*

(ii) *Every vertex of degree 2 outside C is a support and there is exactly one vertex of degree 2 on C or there are exactly two adjacent vertices of degree 2 on C.*

(b) *$e \in E_1^+(G)$ if and only if the following conditions hold.*

(i) *When $\deg u = 2$ the neighbor of u other than v is the only possible non-support vertex of degree 2 and either there is exactly one vertex of* degree 2 on

C or there are exactly two adjacent vertices of degree 2 on C.

(ii) *When $\deg u \geq 3$, either G has a non-support vertex of degree 2 outside C or there exist at least two non-adjacent vertices of degree 2 on C or u is a support vertex of degree 3 to which v is the only pendant neighbor or when u is a support vertex of degree 3 on C, there is a vertex of degree 2 on C that is not a neighbor of u.*

(c) *$e \in E_2^+(G)$ if and only if deg $u = 2$ and there is a non-support vertex of degree 2 outside C that is not adjacent to u or there exist at least two non-adjacent vertices of degree 2 on C.*

Proof. Since e is a pendant edge, $G - e$ is the union of a unicyclic graph G_1 and a single vertex, where $G_1 = G - v$. If l and l_1 are taken to be the number of pendant vertices of G and G_1 respectively, then l_1 is either $l - 1$ or l according as $\deg_G u \geq 3$ or $\deg_G u \geq 2$. Also, by Lemma 2.1 (ii), $Ioc(G \cup K_1)$ is either $l_1 + 2$ or $l_1 + 3$. Therefore $Ioc(G - e)$ is either $l_1 + 2$ or $l_1 + 3$.

(a) Let $e \in E^0(G)$ Then $Ioc(G - e) = Ioc(G) = l + 1$. That is, $Ioc(G \cup K_1) = l_1 + 1$. So, $\deg_G u \geq 3$ and $Ioc(G \cup K_1) = l_1 + 2$. Therefore, by Lemma 2.1 (ii), every vertex of degree 2 outside C in G_1 is a support in G_1 and either there is exactly one vertex of degree 2 on C in G_1 or there are exactly two adjacent vertices of degree 2 on C in G_1. This proves (i) and (ii) of (a).

Conversely, suppose G is an unicyclic graph and e is an edge of G satisfying the conditions (i) and (ii) of (a). Then every vertex with degree 2 outside C in G_1 is a support in G_1 and either there is exactly one vertex of degree 2 on C in G_1 or there are exactly two adjacent vertices of degree 2 on C in G_1 and $l_1 = l - 1$. Hence by Lemma 2.1(ii), $Ioc(G - e) = Ioc(G_1 \cup K_1) = l_1 + 2 = l + 1 = Ioc(G)$ and thus $e \in E^0(G)$.

(b) Suppose $e \in E_1^+(G)$. Then $Ioc(G - e) = l + 2$ so that $Ioc(G \cup K_1) = l_1 + 2$. This leads us to two possibilities namely (1) $Ioc(G \cup K_1) = l_1 + 2$ and $l_1 + l$ and (2) $Ioc(G_1 \cup K_1) = l_1 + 3$ and $l_1 = l - 1$. If Case 1 happens, then $\deg_G u = 2$ and so by Lemma 2.1 (ii), every vertex of degree 2 outside C in G_1 is a support in G_1 and there is exactly one vertex of degree 2 on C in G_1 or there are exactly two adjacent vertices of degree 2 on C in G_1. This proves $b(i)$. Suppose Case 2 happens. Then $\deg_G u \geq 3$ and so again, by Lemma 2.1 (ii), either G_1 has a vertex w with $\deg_{G1} w = 2$ such that w is not a support in G_1 or there exist at least two non-adjacent vertices of degree 2 on C. Note that it is possible that $w = u$; and in this case u is a support in G having v as its only pendant neighbor in G. Further when $w = u$ on C, there must be a vertex of degree 2 on C that is not adjacent to w.

Conversely, assume (i) and (ii) hold true. If $\deg_G u = 2$, then $l_1 = l$. By assumption $b(i)$, the neighbor of u other than v is the only possible non-support vertex of degree 2 in G. So, even if u has such a non-support neighbor w of degree 2 in G, that vertex w will be support (having u as its pendant neighbor) in G_1. Therefore every vertex of degree 2 in G_1 is a support in G_1 and so by Lemma 2.1 (ii), $Ioc(G - e) = Ioc(G_1 \cup K_1) = l_1 + 2 = l_1 + 2$. On the other hand, if $\deg_G u \geq 3$, then $l_1 = l - 1$ and also by assumption that either G_1 has a non-support vertex of degree 2 in G_1 or there exist at least 2 non-adjacent vertices of degree 2 on C so that by Lemma 2.1 (ii), $Ioc(G - e) = Ioc(G_1 \cup K_1) = l_1 + 3 = l + 2$. Thus, in either case $e \in E_1^+(G)$.

(c) Suppose $e \in E_2^+(G)$. Then $Ioc(G - e) = l + 3$ so that $Ioc(G_1 \cup K_1) = l + 3$. That is $Ioc(G_1 \cup K_1) = l_1 + 3$ and $l_1 = l$. In this case $\deg_G u = 2$. Also, by Lemma 2.1 (ii), G_1 has a vertex w with $\deg_{G1} w = 2$ such that w is not a support in G_1 and since u is a pendant in G_1, w is not adjacent to u or there exist at least two non-adjacent vertices of degree 2 on C.

Conversely, assume that $\deg_G u = 2$. Then $l_1 = 1$ and also by assumption either G_1 has a non-support vertex of degree 2 that is not adjacent to u or there exist at least two non-adjacent vertices of degree 2 on C so that by Lemma 2.1 (ii), $Ioc(G - e) = Ioc(G_1 \cup K_1) = l_1 + 3 = l + 3$. Thus $e \in E_2^+(G)$.

Theorem 2.3. *Let G be a unicyclic graph with order $n \geq 3$ with $r < k$. Let $e = uv$ be a non-pendant edge of G. Then*

(a) $e \in E^{-}(G)$ *if and only if both u and v lie in C and are of degree at least* 3.

(b) $e \in E^{0}(G)$ *if and only if the following conditions hold*.

(i) *When e lies on C, exactly one of u and v is of degree* 2.

(ii) *When e lies outside* C, $x \in \{u, v\}$ *is a support if* $\deg x = 2$.

(c) $e \in E_1^{+}(G)$ *if and only if both u and v lie on C and are of degree 2 or exactly one of u and v is a non-support vertex of degree 2 lying outside C.*

(d) $e \in E_1^{+}(G)$ *if and only if both u and v are non-support vertices of degree* 2 *lying outside C.*

Proof. Given that G is unicyclic graph with $r < k$. So, by Theorem 1.2, $Ioc(G) = l + 1$.

(a) Let $e \in E^{-}(G)$. In view of Theorem 1.5 that e cannot be a bridge so that it must lie on C. Certainly, $G - e$ is a tree. Further, if either u or v is of degree 2 in G, then $G - e$ has either $l + 1$ or $l + 2$ pendant vertices so that by Theorem 1.1, $Ioc(G) = l + 1$ or $l + 2$. Therefore, $e \in E^{0}(G) \cup E_1^{+}(G)$, a contradiction. Hence both u and v must be of degree at least 3 in G. Conversely, if $e = uv$ is an edge on C such that both u and v are of degree at least 3, then $G - e$ is a tree with l pendant vertices so that $Ioc(G - e) = l < Ioc(G)$ and therefore $e \in E^{-}(G)$.

(b) Let $e \in E^{0}(G)$. So, when $e = uv$ lies on $C, G - e$ is a tree with $Ioc(G) = l + 1$ so that $G - e$ has $l + 1$ pendant vertices and therefore exactly one of u and v is of degree 2 in G. On the other hand, suppose e is not on C. Certainly $G - e = G_1 \cup T$, where G_1 is a unicyclic graph in which number of vertices on C with degree at least 3 is less than k; and T is a tree. Let l_1 and l_2 denote the number of pendant vertices of G_1 and T respectively. Then by

Theorem 1.4, $Ioc(G-e) = Ioc(G_1) + Ioc(T)$. Now, for instance, suppose u is a vertex of degree 2 such that u is not a support in G. If v is a vertex of degree 2 which is not a support, then $l_1 = l_2 = l + 2$ and $Ioc(G-e) = Ioc(G_1) + Ioc(T)$ $= (l_1 + 1) + l_2 = l + 3 = Ioc(G) + 2$ so that $e \in E_2^+(G)$. If v is a vertex of degree 2 which is a support, then $l_1 = l$ and $Ioc(G-e) = Ioc(G_1) + 1$ $= (l_1 + 1) + 1 = l = 2 = Ioc(G) + 1$ so that $e \in E_1^+(G)$. If $\deg v \geq 3$, then $l_1 + l_2 = l + 1$ and so $Ioc(G-e) = Ioc(G_1) + Ioc(T) = (l_1 + 1) + l_2 = l + 2$ $= Ioc(G) + 1$ so that $e \in E_1^+(G)$. These contradictions prove (ii).

Conversely, if $x \in \{u, v\}$ is a support when $\deg x = 2$ outside C then by Theorem 1.7 (i), $e \in E^0(G)$. If not, then by assumption, exactly one of u and v, say u is of degree 2 on C. Then the vertex u is pendant in $G-e$ and so $Ioc(G-e) - l + 1 = Ioc(G)$, where l denotes the number of pendant vertices of G. Hence $e \in E^0(G)$.

(c) Assume $e \in E_1^+(G)$. Suppose that $e = uv$ is an edge on C. We claim that $\deg u = 2 = \deg v$. Suppose not. Then either exactly one of u and v is of degree 2 or the degree of u and v are at least 3. If the first case happens, then $e \in E^0(G)$, otherwise $e \in E^-(G)$. Hence both u and v are of degree 2. On the other hand, suppose e is not on C. If both u and v are of degree at least 3 in G then $Ioc(G_1) = l_1 + 1$ and $Ioc(T) = l_2$ so that $Ioc(G-e) = l_1 + 1 + l_2 = l + 1$ $= Ioc(G)$ which implies that $e \in E^0(G)$. Hence either u or v is of degree 2. Suppose $\deg_G u = 2 = \deg_G v$. Then by $b(i)$, one of u and v is non-support, but not both; for otherwise $Ioc(G_1) = l_1 + 1$ and $Ioc(T) = l_2$ and so $Ioc(G-e) = l_1 + 1 + l_2 = l + 3$ so that $e \in E_2^+(G)$, a contradiction. Finally, for instance if $\deg_G v = 2$ and $\deg_G u \geq 3$, again by $b(ii)$, v must be non-support.

Conversely, let exactly one of u and v is a non-support vertex of degree 2 outside C. Then by Theorem 1.7 (ii), $e \in E_2^+(G)$. If not, then by assumption, both u and v are of degree 2 on C. Now u and v are pendant vertices in $G-e$

and therefore $Ioc(G - e) = l + 2 = Ioc(G) + 1,$ where l denotes the number of pendant vertices of G. Hence $e \in E_1^+(G)$.

(d) If $e = uv$ is an edge on C then e is not a bridge and hence by Theorem 1.5, $e \notin E_2^+(G)$. Thus, e lies outside C and hence by Theorem 1.7(iii), $e \in E_2^+(G)$ if and only if both u and v are non-support vertices of degree 2 outside C.

Theorem 2.4. *Let G be a unicyclic graph with $r = k$. Let $e = uv$ be a pendant edge of G with $\deg v = 1$. Then*

(a) $e \in E^0(G)$ *if and only if the following conditions hold.*

(i) *When u lies outside C, $\deg u \geq 3$ and when $\deg u = 3$ the vertex u has a pendant neighbor other than v.*

(ii) *When u lies on C, $\deg u \geq 4$ and when $\deg u = 4$ the vertex u has a pendant neighbor other than v.*

(iii) *Every vertex of degree 2 is a support and every vertex of degree 3 on C is a support.*

(b) $e \in E_1^+(G)$ *if and only if the following conditions hold.*

(i) *When $\deg u = 2,$ the neighbor of u other than v is the only possible non-support vertex of degree 2 and every vertex of degree 3 on C is a support.*

(ii) *When u lies outside C with $\deg u = 3,$ either G has a non-support vertex of degree 2 or a non-support vertex of degree 3 on C or u is a support vertex of degree 3 to which v is the only pendant neighbor.*

(iii) *When u lies in C with $\deg u = 3,$ every vertex of degree 2 is a support.*

(iv) *When u lies in C with $\deg u \geq 4,$ either G has a non-support vertex of degree 2 or a non-support vertex of degree 3 on C or u is a support vertex of degree 4 on C to which v is the only pendant neighbor.*

(c) $e \in E_2^+(G)$ *if and only if the following conditions hold.*

(i) *When $\deg u = 2,$ there is a non-support vertex of degree 2 that is not* *adjacent to u or there is a non-support vertex of degree* 3 *on C.*

(ii) *When u lies on C with* $\deg u = 3$, *there is a non-support vertex of degree* 2.

Proof. Let $G - e = G_1 \cup K_1$. Let l and l_1 denote respectively the number of pendant vertices of G and G_1. Then l_1 is either $l - 1$ or l according to $\deg_G u \geq 3$ or $\deg_G u = 2$ Also, by Lemma 2.1 (i), $Ioc(G - e)$ is either $l_1 + 1$ or $l_1 + 2$.

(a) Let $e \in E^0(G)$. Then $Ioc(G - e) = Ioc(G) = l$. That is, $Ioc(G_1 \cup K_1) = l$. Therefore, $\deg_G u \geq 3$ and $Ioc(G_1 \cup K_1) = l_1 + 1$. Hence by Lemma 2.1 (i), every vertex of degree 2 not on C in G_1 is a support in G_1 and every vertex of degree 3 on C in G_1 is a support in G_1. Thus conditions (i), (ii) and (iii) of (a) are true.

Conversely, suppose e satisfies the conditions (i), (ii) and (iii). Then every vertex of degree 2 in G_1 is a support in G_1 and every vertex of degree 3 on C in G_1 is a support in G_1 and $l_1 = l - 1$. Hence by Lemma 2. (i), $Ioc(G - e)$ $= Ioc(G_1 \cup K_1) = l_1 + 1 = l = Ioc(G)$ and thus $e \in E^0(G)$.

(b) Suppose $e \in E_1^+(G)$. Then $Ioc(G - e) = l + 1$ so that $Ioc(G_1 \cup K_1)$ $= l_1 + 1$. This gives rise to only two possibilities namely (1) $Ioc(G_1 \cup K_1)$ $= l_1 + 1$ and $l_1 = l$; (2) $Ioc(G_1 \cup K_1) = l_1 + 2$ and $l_1 = l - 1$. If Case 1 happens, then $\deg_G u = 2$ and so by Lemma 2.1 (i), every vertex of degree 2 in G_1 is a support in G_1 and every vertex of degree 3 on C in G_1 is a support in G_1. This proves $b(i)$. Suppose Case 2 happens. Then $\deg_G u \geq 3$ and so again by Lemma 2.1 (i), G_1 has a vertex w with $\deg_{G1} w = 2$ such that w is not a support in G_1 or G_1 has a vertex x lying on C with $\deg_{G1} x = 3$ such that x is not a support in G_1 (perhaps w or x may be equal to u). If $w = u$ outside C then u is a support in G having v as its only pendant neighbor in G. This proves $b(ii)$. If $w = u$ on C in G, then $\deg_{G1} w = 3$ so that by Lemma 2.1 (ii), every vertex of degree 2 outside C in G_1 is a support in G_1. This completes $b(iii)$. Finally if $x = u$ on C then u is a support in G having v as its only pendant neighbor in G. This completes $b(iv)$.

Conversely, assume that the given conditions (i) − (iv) are true. If $\deg_G u = 2$, then $l_1 = l$ and also by assumption that every vertex of degree 2 in G_1 is a support in G_1 and every vertex of degree 3 on C in G_1 is a support in G_1 so that by Lemma 2.1 (i), $Ioc(G - e) = Ioc(G_1 \cup K_1) = l_1 + 1 = l + 1$. On the other hand, if $\deg_G u \geq 3$, then $l_1 = l - 1$ and also by assumption that G_1 has a non-support vertex of degree 2 in G_1 or a non-support vertex of degree 3 on C in G_1 so that by Lemma 2.1 (i), $Ioc(G - e) = Ioc(G_1 \cup K_1)$ $= l_1 + 2 = l + 1$. Thus, in either case $e \in E_1^+(G)$.

(c) Suppose $e \in E_2^+(G)$. Then $Ioc(G - e) = l + 2$ so that $Ioc(G_1 \cup K_1)$ $= l_1 + 2$. This forces us to have only two possibilities namely (1) $Ioc(G_1 \cup K_1) = l_1 + 2$ and $l_1 = l$ and (2) $Ioc(G_1 \cup K_1) = l_1 + 3$ and $l_1 = l - 1$. If Case 1 happens, then $\deg_G u = 2$, and so by Lemma 2.1 (i), G_1 has a vertex w with $\deg_{G1} w = 2$ such that w is not a support in G_1 or G_1 has a vertex x with $\deg_{G1} x = 3$, such that x is not a support in G_1. This proves $c(i)$. Suppose Case 2 happens, then $\deg_G u \geq 3$ and $l_1 = l - 1$. Certainly $\deg_G u = 3$; for otherwise either $e \in E^0(G)$ or $e \in E_1^+(G)$, a contradiction. Also, if $\deg_G u = 3$ outside C then $e \in E_1^+(G)$ again a contradiction. Thus $\deg_G u = 3$ on C and so by Lemma 2.1 (ii), G_1 has a vertex w with $\deg_{G1} w = 2$ such that w is not a support in G_1. Certainly, w is not adjacent to u. This proves $c(ii)$.

Conversely, assume that conditions (i) and (ii) hold true. If $\deg_G u = 2$ then $l_1 = l$ and also by assumption, there is a non-support vertex of degree 2 in G_1 or there is a non-support vertex of degree 3 on C in G_1 so that by Lemma 2.1 (i), $Ioc(G - e) = Ioc(G_1 \cup K_1) = l_1 + 2 = l + 1$. On the other hand, if $\deg_G u = 3$ on C in G then $\deg_G u = 2$ on C in G_1 and $l_1 = l - 1$. Also, by assumption there is a non-support vertex of degree 2 outside C in G_1. Hence by Lemma 2.1 (ii), $Ioc(G-e) = Ioc(G_1 \cup K_1) = l_1 + 3 = l + 2.$ Thus, in either case $e \in E_2^+(G)$.

Theorem 2.5. *Let G be a unicyclic graph with* $r = k$. *Let* $e = uv$ *be a non-pendant edge of G. Then*

(a) $e \in E^0(G)$ *if and only if when u and v lie outside* C, $x \in \{u, v\}$ *is a support if* $\deg_G x = 2$ *or when u lies on C and v lies outside* C, $\deg_G u \geq 4$ *and v is a support if* $\deg_G v = 2$.

(b) $e \in E_1^+(G)$ *if and only if when u and v lie outside C, exactly one of u and v is a non-support vertex of degree* 2 *or when u lies on C and v lies outside C, either* $\deg_G u = 2$ *or v is a non-support vertex with* $\deg_G v = 2$ *but not both.*

(c) $e \in E_2^+(G)$ *if and only if both u and v are non-support vertices of degree 2 or when u lies on C and v lies outside* C, $\deg_G u = 3$ *and v is a non-support vertex of* $\deg_G v = 2$.

Proof. (a) Let $e \in E^0(G)$. Now when $e = uv$ is an edge on C then $G - e$ is a tree in which $\deg_G u \geq 2$ and $\deg_G v \geq 2$ so that $Ioc(G-e) = l = Ioc(G)$ and hence consider the case that $e = uv$ is an edge outside C. Obviously, $G - e = G_1 \cup T$, where G_1 is a unicyclic graph and T is a tree. Let l_1 and l_2 denote the number of pendant vertices of G_1 and T respectively. Then by Theorem 1.4, $Ioc(G-e) = Ioc(G_1) + Ioc(T)$. Now assume that, both u and v lie outside C. Then the number of vertices on C in G_1 with degree at least 3 is k. Suppose $\deg_G u = 2$ such that u is not a support vertex. If v is not a support vertex with $\deg_G v = 2$, then $l_1 + l_2 = l + 2$ and $Ioc(G-e)$ $= Ioc(G_1) + Ioc(T) = l_1 + l_2 = l + 2 = Ioc(G) + 2$ so that $e \in E_2^+(G)$ If v is a support vertex with $\deg_G v = 2$, then $l_1 = l$ and $Ioc(G-e) = Ioc(G_1)$ $+1 = l_1 + 1 = l + 1 = Ioc(G) + 1$ so that $e \in E_1^+(G)$. then $l_1 + l_2 = l + 1$ and so $Ioc(G-e) = Ioc(G_1) + Ioc(G) + 1 = l_1 + 1 = l + 1 = Ioc(G) + 1$ so that

$e \in E_1^+(G)$. These contradictions prove that $x \in \{u, v\}$ is a support if $\deg x = 2$.

On the other hand, suppose that u lies on C and v lies outside C. We claim that $\deg_G u = 4$ and v is a support when $\deg_G v = 2$. Suppose not. Then either $\deg_G u = 3$ or v is a non-support. If $\deg_G u = 3$, then the number of vertices on C in G_1 with degree at least 3 is $k - 1$ and so $Ioc(G_1) = l_1 + 1$. Suppose that $\deg_G u = 2$. If v is a support in G then $Ioc(G - e) = Ioc(G_1)$ $+1 = l_1 + 1 = l + 1 = Ioc(G) + 1$, implies that $e \in E_1^+(G)$, a contradiction. If v is non-support, then $Ioc(G - e) = Ioc(G_1) + Ioc(T) + 1 = l_1 + 1 + 1 = l + 1$ $= Ioc(G) + 1$, as v is pendant in T, but not in G, again a contradiction. These contradictions show that $\deg_G u \geq 4.$ and v is a support if $\deg_G v = 2$.

Conversely, if both u and v lie outside C such that $x \in \{u, v\}$ is support when $\deg_G x = 2$, then by Theorem 1.7 (i), $e \in E^0(G)$. If not, then by assumption, u lies on C and v lies outside C with $\deg_G u \geq 4.$ and v is support if $\deg_G v = 2$. So in this case $Ioc(G - e) = Ioc(G_1) + 1 = l_1 + 1 = l$ $= Ioc(G) + 1$, and hence $e \in E^0(G)$.

(b) Let $e \in E_1^+(G)$. Suppose that both u and v lie outside C. Then by Theorem 1.7 (ii), exactly one of u and v is a non-support vertex with degree 2. On the other hand, assume u lies on C and v lies outside C. Suppose that $\deg_G u \geq 4$. If $\deg_G v = 3$, then $\deg_{G1} u \geq 3.$ and $\deg_T v \geq 2$ in $G - e$ and so $Ioc(G - e) = l_1 + l_2 = l$, a contradiction. Hence $\deg_G v = 2$. Again, if v is a support, then by (a) $e \in E^0(G)$, a contradiction. Thus, v is a non-support with $\deg_G v = 2$.

Conversely assume that the given conditions are true. We claim that $e \in E_1^+(G)$. Suppose u and v lie outside C with exactly one of u and v is a non-support vertex with degree 2 then by Theorem 1.7 (ii), $e \in E_1^+(G)$. If $\deg_G u = 3$ on C, then $\deg_{G1} u = 3$ which implies that $Ioc(G_1) = l_1 + 1$ so that $Ioc(G - e) = l_1 + 1 + l_2 = l + 1$ and hence $e \in E_1^+(G)$. If not, then v is a non-support vertex with $\deg v = 2$ so that $\deg_{G1} v = 1$. Thus $Ioc(G - e) = l_1 + 1 + l_2 = l + 1$. Hence $e \in E_1^+(G)$.

(c) Let $e \in E_2^+(G)$. Suppose u lies on C and v lies outside C. We claim that $\deg u = 2$ and v is a non-support vertex with $\deg v = 2$. Suppose not. Then by (i) and (ii), either $e \in E^0(G)$ or $e \in E_1^+(G)$, contradiction.

Conversely, when u and v are non-support vertices of degree 2 then by Theorem 1.7 (iii) $e \in E_2^+(G)$. Thus assume that $\deg u = 3$ on C and v is a non-support vertex with $\deg v = 2$ outside C. Then $\deg u = 2$ in G_1 so that $Ioc(G_1) = l_1 + 1$ and $Ioc(G_2) = l_2$ so that Edge-Criticality in Unicyclic graphs upon Outer Connected Independence $Ioc(G - e) = l_1 + 1 + l_2 = l + 2$, as u is not a cut-vertex in G_1 and v is a pendant in G_2. Thus $e \in E_2^+(G)$.

3. Unicyclic Graphs with $E(G) = E^0(G)$

It has been observed in [6], that there are graphs G for which $E(G) = E^0(G)$ or $E(G) = E_1^+(G)$ whereas there is no graph G with $E(G) = E_2^+(G)$ or $E(G) = E^-(G)$ In the following theorems, we characterize unicyclic graphs for which $E(G) = E^0(G)$, when $r = k$ and prove that there is no unicyclic graph G with $E(G) = E^0(G)$, when $r < k$. Also prove that there is no unicyclic graph G other than the cycle with $E(G) = E_1^+(G)$.

Theorem 3.1. *There is no unicyclic graph other than the cycle for which* $E(G) = E_1^+(G)$.

Proof. Suppose that there exist a unicyclic graph $G \not\cong C$ such that $E(G) = E_1^+(G)$. Choose an arbitrary edge e on the cycle C. If $r = k$, then by Theorem 2.5 (a), we get $e \in E^-(G)$ or $e \in E^0(G)$, a contradiction. If not, then $r < k$. Now choose an edge incident with a vertex of degree at least 3. Then by Theorem 2.3 (a) and (b) either $e \in E^-(G)$ or $e \in E^0(G)$, again a contradiction. Hence the proof.

Theorem 3.2. *Let G be a unicyclic with the cycle C. Then* $e \in E^0(G)$ *if and only if* $r = k$ *and the following conditions hold*.

(i) *G has neither a vertex of degree* 2 *nor a vertex of degree* 3 *lying on C*.

(ii) *Every support vertex of degree 3 lying outside C and every support vertex of degree* 4 *lying on C has at least two pendant neighbors*.

Proof. Suppose $E(G) = E^0(G)$. We first prove that $r = k$. Suppose $r < k$. Certainly $G \not\cong C$, since every edge of C belongs to $E_1^+(G)$. Hence G has at least one vertex of degree greater than or equal to 3 on C. Also, since G has a pendant edge and by assumption that pendant edge belongs to $E^0(G)$, by Theorem 2.2 a (ii) there is exactly one vertex of degree 2 on C or there are exactly two adjacent vertices of degree 2 on C. If there exist two adjacent vertices, say u and v of degree 2 on C, by Theorem 2.3 (c) $e \in E_1^+(G)$, as e is non-pendant, a contradiction. Hence G has exactly one vertex of degree 2 on C. Now choose an edge $e = xy$ on C with $\deg x \geq 3$ and $\deg y \geq 3$ (this is possible as there are at least 3 vertices on the cycle C). Now by the same Theorem 2.3 (a), the edge $e = xy \in E^-(G)$, again a contradiction. Hence $r = k$.

Let u be an arbitrary vertex of G with $\deg u \geq 2$. Suppose that u is a support and $e = uv$ is an edge with $\deg v = 1$. Since $e \in E^0(G)$, by Theorem 2.4 $a(i - ii)$, $\deg u \geq 3$ when u lies outside C and $\deg u \geq 4$ when u lies on C. On the other hand, suppose that the vertex u is not support. Then every $x \in N(u)$ is not pendant and by assumption $e = ux \in E^0(G)$, so that by Theorem 2.5 (a), $\deg u \geq 3$ when u lies outside C and $\deg u \geq 4$ when u lies on C. This proves (i). Further if w is either a support vertex of degree 3 lying outside C or a support vertex of degree 4 lying on C and $e = wv$ is a pendant edge, then by assumption and by Theorem 2.4 $a(i - ii)$ the vertex w must have a pendant neighbor other than v. This completes (ii).

Conversely, suppose that the unicyclic graph with $r = k$ satisfies the given condition (i) and (ii). Let $e = uv$ be an arbitrary edge of G. If e is not a pendant edge then by assumption, for $x \in \{u, v\}$ either $\deg x \geq 3$ or $\deg x \geq 4$ according as the vertex x lies on C or lies outside C. Hence by Theorem 2.5 (a), $e \in E^0(G)$. On the other hand, when e is a pendant edge with $\deg v = 1$, if either $\deg x = 3$ outside C or $\deg x = 4$ on C, by condition (ii), the vertex u will have pendant neighbor other than v and thus by Theorem 2.4 $a(i - ii)$, $e \in E^0(G)$. Therefore $E(G) = E^0(G)$.

Every edge of the unicyclic graph G of Figure 3.1, satisfies the conditions (i) and (ii) of Theorem 3.2 and hence $E(G) = E^0(G)$.

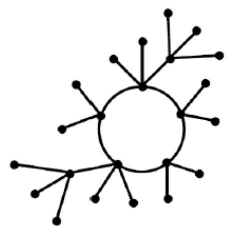

Figure 3.1. A unicyclic graph G with $E(G) = E^0(G)$.

4. Conclusion and Scope

The study of the effect of the removal of a vertex or an edge on any graph theoretic parameter has interesting applications in the context of network. In this chapter, we have initiated a study on the effect of the outer connected independence number upon the edge removal and we have proved that, for a graph G and an edge e of G, the value of $Ioc(G - e)$ increases by at most two and decreases by at most one. Further we have completely determined the properties of the sets $E^-(G)$, $E^0(G)$, $E_1^+(G)$ and $E_2^+(G)$ for paths, complete multipartite graphs, wheels, trees and unicyclic graphs. The following are some interesting problems for further research on edge criticality with respect to the outer connected independence in graphs.

(A) In [6], we have proved that there is no graph G with $E(G) = E-(G)$ or $E(G) = E_1^+(G)$, whereas we have observed that there are graphs G for which $E(G) = E^0(G)$, or $E(G) = E_1^+(G)$. Further, this is completely settled for trees and unicyclic graphs.

Now one can characterize the graphs G for which $E(G) = E^0(G)$ and $E(G) = E_1^+(G)$.

(B) Determination of the sets $E^-(G), E^0(G), E_1^+(G)$ and $E_2^+(G)$ for graphs with diameter two is an interesting problem.

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