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Non-negative Majority Total Domination In Graphs

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Abstract. A two-valued function $f : V \to \{-1, +1\}$ defined on the vertices of a graph $G = (V, E)$, is a *non-negative majority total dominating function* if the sum of its function values over at least half the open neighbourhood is at least zero. That is, for at least half of the vertices $v \in V$, $f(N(v)) \geq 0$, where $N(v)$ consists of every vertex adjacent to v. The *non-negative majority total domination number* of a graph G, denoted $\gamma^{Nt}_{maj}(G)$, is the minimum value of $\sum\limits_{v \in V(G)} f(v)$ over all non-negative majority total dominating functions f of G. In this paper, we initialize the study of non-negative majority total domination in graphs.

1 Introduction

By a graph $G = (V, E)$, we mean a finite, non-trivial, connected, and undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartand and Lesniak [1].

The study of domination is one of the fastest growing areas within graph theory. A subset D of vertices is said to be a *dominating set* of G if every vertex in V either belongs to D or is adjacent to a vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. Survey of several advanced topics on domination are given in the book edited by Haynes et al [2].

For a real valued function $f : V \to R$ on V, weight of f is defined to be $w(f) = \sum\limits_{v \in V} f(v)$ and also for a subset $S \subseteq V$, we define $f(S) = \sum\limits_{v \in S} f(v)$. Therefore $w(f) = f(V)$. Majority domination was first introduced by Broere et al. in [3] and further studied in [4, 5].

A function $f : V \to \{-1, +1\}$ is called a *signed majority total dominating function* if $f(N(v)) \geq 1$ for at least half of the vertices in graph G. The *signed majority total domination number* of G, is denoted by $\gamma^t_{maj}(G)$ and is defined as
$\gamma^t_{maj}(G)$=$min\,\{w(f) \mid f\ is\ a\ signed\ majority\ total\ dominating function\ of\ G\}$. Further, the concept of non-negative signed domination of a graph was introduced in [6]. In this paper, we initiate the study of non-negative majority total domination in graphs.

2 Common Classes of Graphs

Definition 2.1. A function $f : V \to \{-1, +1\}$ is called a *non-negative majority total dominating function* (briefly NMTDF) if $f(N(v)) \geq 0$ for at least half of the vertices in G. The *non-negative majority total domination number* of G, denoted by $\gamma^{Nt}_{maj}(G)$, is defined as $\gamma^{Nt}_{maj}(G)$= $min\,\{w(f) \mid f\ is\ a\ NMTDF\ of\ G\}$.

Let us follow throughout the paper the following terminologies.
If f is a non-negative majority total dominating function of a graph G, then we define the sets P_f,M_f and N_f as follows.

(i) $P_f(G) = \{v \in V(G) : f(v) = 1\}$

(ii) $M_f(G) = \{v \in V(G) : f(v) = -1\}$

(iii) $N_f(G) = \{v \in V(G) : f(N(v)) \geq 0\}$

Theorem 2.2. *For any path P_n on $n \geq 2$ vertices,*

$$\gamma^{Nt}_{maj}(P_n) = 2\left\lceil \frac{n}{4} \right\rceil - n$$

Proof. Let $P_n = (v_1, v_2, ..., v_n)$ and let f be a non-negative majority total dominating function of P_n. Then for any vertex $v \in N_f$, at least one neighbour of v belongs to P_f. Since $|N_f| \geq \left\lceil \frac{n}{2} \right\rceil$, we have $|P_f| \geq \left\lceil \frac{n}{4} \right\rceil$ which implies that $|M_f| \leq n - \left\lceil \frac{n}{4} \right\rceil$. Hence $|P_f| - |M_f| \geq 2\left\lceil \frac{n}{4} \right\rceil - n$.

On the other hand, define the function $g : V \to \{-1, +1\}$ by

$$g(v_i) = \begin{cases} +1 & \text{if } 2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{4} \right\rceil \text{ } and \text{ } i \equiv 2(mod\ 3) \\ -1 & \text{otherwise} \end{cases}$$

Then we can verify that $g(N(v)) \geq 0$ for at least half of the vertices in G with weight $2\left\lceil \frac{n}{4} \right\rceil - n$. Hence $\gamma^{Nt}_{maj}(P_n) \leq w(f) = 2\left\lceil \frac{n}{4} \right\rceil - n$. Consequently, the result follows. □

Corollary 2.3. *For any negative integer k, there exists a graph G for which $\gamma^{Nt}_{maj}(G) = k$.*

Theorem 2.4. *For $n \geq 3$, an integer $\gamma^{Nt}_{maj}(C_n) = \gamma^{Nt}_{maj}(P_n)$.*

Proof. Let $C_n = (v_1, v_2, ..., v_n)$ be the cycle on n vertices. Then $C_n - v_1v_n$ is a path on n vertices and also the function g defined on $P_n = C_n - v_1v_n$ as in Theorem 2.2, would be a non-negative majority total domination for the cycles C_n so that $\gamma^{Nt}_{maj}(C_n) \leq \gamma^{Nt}_{maj}(P_n)$. We now show that $\gamma^{Nt}_{maj}(C_n) \geq \gamma^{Nt}_{maj}(P_n)$. Let f be a minimum non-negative majority total domination of C_n. For $n \geq 3$, by Theorem 2.2, $\gamma^{Nt}_{maj}(P_n) < 0$. Therefore, $|P_f| - |M_f| = f(V) = \gamma^{Nt}_{maj}(C_n) \leq \gamma^{Nt}_{maj}(P_n) < 0$ which in turn implies that $|M_f| > |P_f|$. This means that M_f must contain two adjacent vertices v_i, v_j. Consider now the path P on n vertices obtained from C_n by removing the edge v_iv_j. The number of non-negative open neighborhood sums under f on P is the same as that of f on C_n. It follows that f is a non-negative majority total dominating function of P and hence $\gamma^{Nt}_{maj}(P_n) = \gamma^{Nt}_{maj}(P) \leq f(V) = \gamma^{Nt}_{maj}(C_n)$. □

Theorem 2.5. *For any complete graph $K_n (n \geq 2)$, we have*

$$\gamma^{Nt}_{maj}(K_n) = \begin{cases} -1 & \text{if } n \text{ } is \text{ } odd \\ 0 & otherwise \end{cases}$$

Proof. Let f be a non-negative majority total dominating function of K_n. Then $|P_f| + |M_f| = n$ and $|P_f| - |M_f| = f(V)$. Now, consider a vertex v of K_n with $f(N(v)) \geq 0$. Certainly, $f(V) = f(N(v)) + f(v) \geq 0 - 1$ which means that $|P_f| - |M_f| \geq -1$. It follows that $|P_f| \geq \left\lceil \frac{n-1}{2} \right\rceil$ and $|M_f| \leq \left\lfloor \frac{n+1}{2} \right\rfloor$. Thus $\gamma^{Nt}_{maj}(K_n) \geq \left\lceil \frac{n-1}{2} \right\rceil - \left\lfloor \frac{n+1}{2} \right\rfloor$. That is, $\gamma^{Nt}_{maj}(K_n) \geq \begin{cases} -1 & \text{if } n \text{ } is \text{ } odd \\ 0 & \text{otherwise} \end{cases}$

Now, consider the function $g : V \to \{-1, +1\}$ that assigns the value -1 for $\left\lceil \frac{n}{2} \right\rceil$ vertices of K_n and the value +1 for the remaining vertices. Obviously, g is a non-negative majority total dominating function of K_n, so that $\gamma^{Nt}_{maj}(K_n) \leq n - 2\left\lceil \frac{n}{2} \right\rceil$.

That is, $\gamma^{Nt}_{maj}(K_n) \leq \begin{cases} -1 & \text{if } n \text{ } is \text{ } odd \\ 0 & \text{otherwise} \end{cases}$ □

Theorem 2.6. *For any complete bipartite graph $K_{r,s} (s \geq r \geq 1)$*

$$\gamma^{Nt}_{maj}(K_{r,s}) = \begin{cases} -s & \text{if } r \text{ } is \text{ } even \\ 1 - s & \text{if } r \text{ } is \text{ } odd \end{cases}$$

Proof. Let (U, W) be the bipartition of $K_{r,s}$ with $|U| = r$ and $|W| = s$. Let f be a minimum non-negative majority total dominating function of $K_{r,s}$. Then W contains a vertex x with $f(N(x)) \geq 0$ when $r < s$. Certainly, when $r = s$, either U or W contains such a vertex x. Without loss of generality assume that W contains such a vertex x. This implies that $f(U) \geq 0$. If U^+ and U^- denote the set of vertices that are assigned with +1 and -1 respectively, then $f(U) = |U^+| - |U^-|$ so that $|U^+| - |U^-| \geq 0$. Obviously, $|U^+| + |U^-| = r$. Using these, we get $|U^+| \geq \lceil \frac{r}{2} \rceil$ and $|U^-| \leq \lfloor \frac{r}{2} \rfloor$ and consequently $f(U) \geq \lceil \frac{r}{2} \rceil - \lfloor \frac{r}{2} \rfloor$.

We now claim that every vertex of W receives the value -1 under f. If not, there exists a vertex $w \in W$ with $f(w) = +1$. Now the function $g : V(K_{r,s}) \to \{-1, +1\}$ obtained from f by replacing $f(w)$ by -1, is a non-negative majority total dominating function with $w(g) = w(f) - 2$, which is a contradiction to the minimality of f. Hence every vertex of W receives -1 under f so that $f(W) = -s$. Thus $f(V) = f(U) + f(W) \geq \lceil \frac{r}{2} \rceil - \lfloor \frac{r}{2} \rfloor - s$. That is,

$$\gamma_{maj}^{Nt}(K_{r,s}) \geq \begin{cases} -s & \text{if } r \textit{ is even} \\ 1 - s & \text{if } r \textit{ is odd} \end{cases}$$

Now, the function that assigns the value +1 to $\lceil \frac{r}{2} \rceil$ vertices of U and the value -1 for the remaining vertices of $K_{r,s}$ is a non-negative majority total dominating function of $K_{r,s}$ with weight $\lceil \frac{r}{2} \rceil - \lfloor \frac{r}{2} \rfloor - s$. This proves the result. □

3 Bounds

In this section, we discuss some bounds for the non-negative majority total domination.

Theorem 3.1. *A NMTDF f on a graph G is minimal only if for every vertex $v \in V$ with $f(v) = 1$, there exists a vertex $u \in N(v)$ with $f(N(u)) \in \{0, 1\}$.*

Proof. Let f be a minimal NMTDF and assume that there is a vertex v with $f(v) = 1$ and $f(N(u)) \notin \{0, 1\}$ for every vertex $u \in N(v)$. Now, define a new function $g : V \to \{-1, +1\}$ by $g(v) = -1$ and $g(w) = f(w)$ for all $w \neq v$. Then for all $u \in N(v)$, either $f(N(u)) \leq -1$, in which case $g(N(u)) = f(N(u)) - 2 \leq -3$ or $f(N(u)) \geq 2$, in which case $g(N(u)) = f(N(u)) - 2 \geq 0$. For $w \notin N(v)$, we have $g(N(w)) = f(N(w))$. Thus $|N_g| = |N_f|$ and so g is an NMTDF on G. Since $w(g) < w(f)$, the minimality of f is contradicted. □

Theorem 3.2. *Let G be a graph with the degree sequence $(d_1, d_2, ..., d_n)$ such that $d_1 \leq d_2 \leq ... \leq d_n$. Then $\gamma_{maj}^{Nt}(G) \geq -n + \frac{2}{d_n} \sum_{j=1}^{\lceil \frac{n}{2} \rceil} \left\lfloor \frac{d_j}{2} \right\rfloor$.*

Proof. Let g be a non-negative majority total dominating function of G. Then $g(N(v)) \geq 0$ for at least half of the vertices say $v_{j1}, v_{j2}, ..., v_{j\lceil \frac{n}{2} \rceil}$ with corresponding degrees $d_{j1}, d_{j2}, ..., d_{j\lceil \frac{n}{2} \rceil}$ respectively in G. Let $f(x) = \frac{(g(x)+1)}{2}$ for all vertices in G. Then f is a 0-1 valued function.First,

$$\begin{aligned}\sum_{i=1}^{\lceil \frac{n}{2} \rceil} f(N(v_{ji})) &= \sum_{i=1}^{\lceil \frac{n}{2} \rceil} \frac{g(N(v_{ji})) + d_{ji}}{2} \\ &\geq \sum_{i=1}^{\lceil \frac{n}{2} \rceil} \frac{d_{ji}}{2} \\ &\geq \sum_{j=1}^{\lceil \frac{n}{2} \rceil} \frac{d_j}{2}\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{i=1}^{\lceil \frac{n}{2} \rceil} f(N(v_{ji})) &\leq \sum_{j=1}^{n} f(N(v_j)) \\ &= \sum_{j=1}^{n} deg\ v_j\ f(v_j) \\ &\leq d_n\ f(V)\end{aligned}$$

Therefore, $f(V) \geq \frac{1}{d_n} \sum_{j=1}^{\lceil \frac{n}{2} \rceil} \frac{d_j}{2}$. Also since $\gamma_{maj}^{Nt}(G) = g(V) = 2f(V) - n$, we have $\gamma_{maj}^{Nt}(G) \geq -n + \frac{2}{d_n}\sum_{j=1}^{\lceil \frac{n}{2} \rceil} \left\lfloor \frac{d_j}{2} \right\rfloor$. □

Theorem 3.3. *If G is a graph of order n, then*

$$\gamma_{maj}^{Nt}(G) \geq \begin{cases} \frac{n\delta - 2n\Delta}{\Delta+\delta} & \text{if } n \text{ is even} \\ \frac{n\delta+\Delta(1-2n)}{\Delta+\delta} & \text{if } n \text{ is odd} \end{cases}$$

Proof. Let f be a $\gamma_{maj}^{Nt}(G)$-function on G. Let $P_f = P_\Delta \cup P_\delta \cup P_\ominus$ where P_Δ and P_δ are sets of all vertices of P_f with degree equal to Δ and δ, respectively, and $P_\ominus$ contains all other vertices in P_f. Let $M_f = M_\Delta \cup M_\delta \cup M_\ominus$ where M_Δ, M_δ and $M_\ominus$ are defined similarly. Further, for $i \in \{\Delta, \delta, \ominus\}$, let V_i be defined by $V_i = P_i \cup M_i$. Thus, $n = |V_\Delta| + |V_\delta| + |V_\ominus|$.

Since for at least half of the vertices $v \in V$, $f(N(v)) \geq 0$, we have

$$\sum_{v \in V} f(N(v)) \geq 0 \left\lceil \frac{n}{2} \right\rceil - \Delta(n - \left\lceil \frac{n}{2} \right\rceil) = \Delta(\left\lceil \frac{n}{2} \right\rceil - n).$$

The sum $\sum_{v\in V} f(N(v))$ counts the value $f(v)$ exactly $deg\ v$ times for each vertex $v \in V$. That is $\sum_{v\in V} f(N(v)) = \sum_{v \in V} f(v)\ deg\ v$.

Thus $\sum_{v\in V} f(v)\ deg\ v \geq \Delta(\left\lceil \frac{n}{2} \right\rceil - n)$.

By spliting the sum up into the six summations and replacing $f(v)$ with the corresponding value of +1 or -1 yields

$$\sum_{v\in P_\Delta} deg\ v + \sum_{v\in P_\delta} deg\ v + \sum_{v\in P_\ominus} deg\ v - \sum_{v\in M_\Delta} deg\ v - \sum_{v\in M_\delta} deg\ v - \sum_{v\in M_\ominus} deg\ v \geq \Delta(\left\lceil \frac{n}{2} \right\rceil - n).$$

We know that $deg\ v = \Delta$ for all $v \in \{P_\Delta, M_\Delta\}$ and $deg\ v = \delta$ for all $v \in \{P_\delta, M_\delta\}$. Also, for any vertex $v \in \{P_\ominus, M_\ominus\}$, $\delta + 1 \leq deg\ v \leq \Delta - 1$.

Therefore, we have

$$\Delta |P_\Delta| + \delta |P_\delta| + (\Delta - 1) |P_\ominus| - \Delta |M_\Delta| - \delta |M_\delta| - (\delta + 1) |M_\ominus| \geq \Delta(\left\lceil \frac{n}{2} \right\rceil - n).$$

For $i \in \{\Delta, \delta, \ominus\}$, we replace $|P_i|$ with $|V_i| - |M_i|$ in the above inequality, we have

$$\Delta |V_\Delta| + \delta |V_\delta| + (\Delta - 1) |V_\ominus| \geq \Delta(\left\lceil \frac{n}{2} \right\rceil - n) + 2\Delta |M_\Delta| + 2\delta |M_\delta| + (\Delta + \delta) |M_\ominus|.$$

It follows that

$(2n - \left\lceil \frac{n}{2} \right\rceil)\Delta \geq 2\Delta |M_\Delta| + 2\delta |M_\delta| + (\Delta + \delta) |M_\ominus| + (\Delta - \delta) |V_\delta| + |V_\ominus|$

$= 2\Delta |M_\Delta| + 2\delta |M_\delta| + (\Delta + \delta) |M_\ominus| + (\Delta - \delta)(|P_\delta| + |M_\delta|) + (|P_\ominus| + |M_\ominus|)$

$= 2\Delta |M_\Delta| + (\Delta + \delta) |M_\delta| + (\Delta + \delta + 1) |M_\ominus| + (\Delta - \delta) |P_\delta| + |P_\ominus|$

$\geq (\Delta + \delta) |M_\Delta| + (\Delta + \delta) |M_\delta| + (\Delta + \delta) |M_\ominus| = (\Delta + \delta) |M_f|$.

Therefore, $|M_f| \leq \frac{(2n - \lceil \frac{n}{2} \rceil)\Delta}{\Delta+\delta}$.

Hence, $\gamma_{maj}^{Nt}(G) = |P_f| - |M_f| = n - 2 |M_f| \geq n - 2\frac{(2n - \lceil \frac{n}{2} \rceil)\Delta}{\Delta+\delta}$ □

Theorem 3.4. *Let G be a graph of order n and let k be any integer. Then $\gamma_{maj}^{Nt}(G) = k$ if and only if there exists a partition (P_f, M_f) of V for which*

(i) $|N(x) \cap P_f| - |N(x) \cap M_f| \geq 0$ *for at least half of the vertices of G.*

(ii) $|P_f| - |M_f| = k$.

(iii) For any $P' \subseteq P_f$ and any $M' \subseteq M_f$ satisfying $\left|P'\right| > \left|M'\right|$, we have
$\left|\left\{x \in V \mid 2(\left|N(x) \cap P'\right| - \left|N(x) \cap M'\right|) > |N(x) \cap P_f| - |N(x) \cap M_f|\right\}\right| > n - \left\lceil \frac{n}{2} \right\rceil$.

Proof. Suppose $\gamma^{Nt}_{maj}(G) = k$. Let f be a NMTDF of G such that $f(V) = \gamma^{Nt}_{maj}(G) = k$. Then (P_f, M_f) constitutes a partition of V. For each $x \in N_f$, $|N(x) \cap P_f| - |N(x) \cap M_f| \geq 0$. Since $|N_f| \geq \lceil \frac{n}{2} \rceil$, condition (i) holds. Since $f(V) = |P_f| - |M_f|$, condition (ii) holds. To verify condition (iii), consider any $P^{'} \subseteq P_f$ and $M^{'} \subseteq M_f$ such that $\left|P^{'}\right| > \left|M^{'}\right|$. Let $X = (P_f - P^{'}) \cup M^{'}$ and $Y = (M_f - M^{'}) \cup P^{'}$. Now, define a function $g : V \to \{-1, +1\}$ by $g(x) = 1$ for every $x \in X$ and $g(x) = -1$ for every $x \in Y$. Then

$$\begin{aligned} g(V) &= |X| - |Y| \\ &= (|P_f| - \left|P^{'}\right| + \left|M^{'}\right|) - (|M_f| - \left|M^{'}\right| + \left|P^{'}\right|) \\ &= |P_f| - |M_f| - 2(\left|P^{'}\right| - \left|M^{'}\right|) \\ &< |P_f| - |M_f| \\ &= f(V) = \gamma^{Nt}_{maj}(G). \end{aligned}$$

Thus g is not a NMTDF of G and hence $|N_g| < \lceil \frac{n}{2} \rceil$. Consequently, $|\{x \in V | g(N(x)) < 0\}| = |V - N_g| = n - |N_g| > n - \lceil \frac{n}{2} \rceil$. Also,

$$\begin{aligned} g(N(x)) &= |N(x) \cap X| - |N(x) \cap Y| \\ &= |N(x) \cap P_f| - |N(x) \cap M_f| - 2(\left|N(x) \cap P^{'}\right| - \left|N(x) \cap M^{'}\right|). \end{aligned}$$

Hence we obtain condition(iii).

For the sufficiency, suppose there is a partition (P_f, M_f) of V such that conditions (i), (ii) and (iii) hold. Define a function $f : V \to \{-1, +1\}$ by $f(x) = 1$ for every $x \in P_f$ and $f(x) = -1$ for every $x \in M_f$. Then by condition(i), $f(N(x)) = |N(x) \cap P_f| - |N(x) \cap M_f| \geq 0$ for at least half vertices of G. Thus f is NMTDF of G so that by condition(ii) $\gamma^{Nt}_{maj}(G) \leq |P_f| - |M_f| = k$. We now show that $\gamma^{Nt}_{maj}(G) \geq |P_f| - |M_f|$. Suppose to the contrary, $\gamma^{Nt}_{maj}(G) < |P_f| - |M_f|$. Let g be a NMTDF of G such that $\gamma^{Nt}_{maj}(G) = g(V)$. Let $X = \{x \in V | g(x) = 1\}$ and $Y = \{x \in V | g(x) = -1\}$. Let $P^{'} = P_f - X$ and $M^{'} = M_f - Y$. Then $P^{'} \subseteq P_f$, $M^{'} \subseteq M_f$, $X = (P_f - P^{'}) \cup M^{'}$ and $Y = (M_f - M^{'}) \cup P^{'}$. Moreover,

$$\begin{aligned} |P_f| - |M_f| + 2(\left|M^{'}\right| - \left|P^{'}\right|) &= |P_f| - \left|P^{'}\right| + \left|M^{'}\right| - |M_f| + \left|M^{'}\right| - \left|P^{'}\right| \\ &= |X| - |Y| = \gamma^{Nt}_{maj}(G) \\ &< |P_f| - |M_f|, \text{ so that } \left|P^{'}\right| > \left|M^{'}\right|. \end{aligned}$$

Therefore by condition (iii),
$|V - N_g| = |\{x \in V | g(N(x)) < 0\}|$
$= \left|\left\{x \in V | 2(\left|N(x) \cap P^{'}\right| - \left|N(x) \cap M^{'}\right|) > |N(x) \cap P_f| - |N(x) \cap M_f|\right\}\right| > n - \lceil \frac{n}{2} \rceil$. Thus, $|N_g| < \lceil \frac{n}{2} \rceil$, contradicting the fact that g is NMTDF of G. Hence, $\gamma^{Nt}_{maj}(G) \geq |P_f| - |M_f|$. □

4 Trees

In this section, we determine upper bound of non-negative majority total domination of a tree. By assigning $+1$ to the center of a star and -1 to all the leaves we obtain a NMTDF of the star. Thus

Proposition 4.1. *For $n \geq 3$, $\gamma^{Nt}_{maj}(K_{1,n-1}) = 2 - n$.*

Hence the Non-negative majority total domination number of a tree can be arbitrarily large negative.

Theorem 4.2. *For any tree T of order $n \geq 2$, $\gamma^{Nt}_{maj}(T) \leq 2 \lceil \frac{n}{4} \rceil - n$.*

Proof. We proceed by induction on the order $n \geq 2$ of a tree T. If $n \in \{2, 3\}$, then $T = P_n$ and the result follows from Theorem 2.2. This proves the base cases when $n = 2$ or $n = 3$. For $n \geq 4$, assume that every nontrivial tree $T^{'}$ of order $n^{'} < n$, $\gamma^{Nt}_{maj}(T^{'}) \leq 2 \left\lceil \frac{n^{'}}{4} \right\rceil - n^{'}$. Let T be a tree of order n. If T is a star, then by Proposition 4.1, $\gamma^{Nt}_{maj}(T) = 2 - n \leq 2\left\lceil \frac{n}{4} \right\rceil - n$. Hence the desired result follows if T is a star. Thus we assume that $diam(T) \geq 3$.

Let T be rooted at a leaf r of a longest path. Let v be a vertex at distance $diam(T) - 1$ from r on a longest path starting at r and let w be the parent of v. Let $|N(v) - \{w\}| = m$. Then $m \geq 1$. Let $T^{'} = T - (N(v) - \{w\})$. Then $T^{'}$ has order $n^{'} = n - m$. Since $diam(T) \geq 3$, we have $n^{'} \geq 2$. Let $f^{'}$ be a $\gamma^{Nt}_{maj}(T^{'})$ - function. Let $f : V \to \{-1, +1\}$ be the function defined by $f(u) = -1$ for every child of v and every vertex whose open neighborhood sum is at least zero in $T^{'}$ also has open neighborhood sum at least zero in T, while each child of v has $f(N(u)) \geq 0$. Hence $\left\lceil \frac{n^{'}}{2} \right\rceil + m \geq \left\lceil \frac{n}{2} \right\rceil$ vertices of T has open neighborhood sum at least zero and so f is a NMTDF of T. Thus $\gamma^{Nt}_{maj}(T) \leq f(V(T)) = f^{'}(V(T^{'}) - m$. By the inductive hypothesis, $\gamma^{Nt}_{maj}(T^{'}) \leq 2\left\lceil \frac{n^{'}}{4} \right\rceil - n^{'} = 2\left\lceil \frac{n-m}{4} \right\rceil - n + m$ and so $\gamma^{Nt}_{maj}(T) \leq 2\left\lceil \frac{n-m}{4} \right\rceil - n + m - m$. Since $m \geq 1$, $\gamma^{Nt}_{maj}(T) \leq 2\left\lceil \frac{n-1}{4} \right\rceil - n \leq 2\left\lceil \frac{n}{4} \right\rceil - n$. Hence the desired result follows. □

As an immediate consequence of Theorem 3.4, we have the following result.

Corollary 4.3. *Let T be a tree of order n. Then $\gamma^{Nt}_{maj}(T) = 2\left\lceil \frac{n}{4} \right\rceil - n$ if and only if there exists a partition (P_f, M_f) of V for which*

(i) $|N(x) \cap P_f| - |N(x) \cap M_f| \geq 0$ *for at least half of the vertices of T.*

(ii) $|P_f| - |M_f| = 2\left\lceil \frac{n}{4} \right\rceil - n$.

(iii) For any $P^{'} \subseteq P_f$ and any $M^{'} \subseteq M_f$ satisfying $\left|P^{'}\right| > \left|M^{'}\right|$, we have
$$\left|\left\{x \in V \mid 2(\left|N(x) \cap P^{'}\right| - \left|N(x) \cap M^{'}\right|) > |N(x) \cap P_f| - |N(x) \cap M_f|\right\}\right| > n - \left\lceil \frac{n}{2} \right\rceil.$$

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