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COMMON FIXED POINT RESULTS IN GENERALIZED FUZZY METRIC SPACES AND OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS

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Abstract

In this paper, we introduce a new concept of common fixed point theorem in generalized fuzzy metric spaces and occasionally weakly compatible mappings.

1. Introduction

The fixed point theory has been studied and generalized in different spaces. Fuzzy set theory is one of uncertainty approaches where in topological structure are basic tools to develop mathematical models compatible to concrete real life situation. Fuzzy set was defined by Zadeh [27]. Kramosil and Michalek [15] introduced fuzzy metric space, George and Veermani [7] modified the notion of fuzzy metric spaces with the help of continuous *t*-

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norms. Many researchers have obtained common fixed point theorem for mapping satisfying different types of commutativity conditions. Jain and Singh [29] proved a fixed point theorem for six self maps in a fuzzy metric space. In this paper, a fixed point theorem for six self maps has been established using the concept of weak compatibility of pairs of self maps in fuzzy metric space. New concept of common fixed point theorem in generalized fuzzy metric spaces and occasionally weakly compatible mappings is introduced.

2. Preliminaries

Definition 2.1. A 3-tuple $(X, M, *)$ is called M-fuzzy metric space if X is an arbitrary non empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions: for each $x, y, z, \in X$ and $M(x, y, z, t) > 0$,

(M1) $M(x, y, z, t) > 0$,

(2) $M(x, y, z, t) > 0$, if and only if $M(x, y, z, t) > 0$,

(M3) $M(x, y, z, t) = (p\{x, y, z\}, t)$, when p is the permutation function.

(M4) $M(x, y, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s)$,

(M5) $M(x, y, z, \cdot) : (0, \infty) \to [0, 1]$ is continuous,

(M6) $\lim_{t \to \infty} M(x, y, z) = 1$ for all $x, y, z \in X$.

Definition 2.2. Two mappings f and g of a generalized fuzzy metric space $(X, M, *)$ into itself are said to be weakly commuting if

$$M(fgx, fx, fx) \geq M(fx, gx, gx, t), \text{ for all } x \in X \text{ and } t > 0.$$

Definition 2.3. Two mappings f and g of a generalized fuzzy metric space $(X, M, *)$ into itself are R-weakly commuting provided there exists some positive real number R such that

$$M(fgx, gfx, t) \geq M\left(fx, gx, gx, \frac{t}{R}\right), \text{ for all } x \in X, R > 0 \text{ and } t > 0$$

Remark 2.4. Clearly point R-weakly commutativity implies weak commutativity only when $R \leq 1$.

Definition 2.5. Two self maps *f* and *g* of a generalized fuzzy metric space $(X, M, *)$ are called reciprocally continuous on X if $fgx_n = fx$ and $gfx_n = gx$, whenever $\{x_n\}$ is a sequence in X such that $fx_n = gx_n = x$ or some x in X.

Definition 2.6. Two self mappings *f* and *g* of a generalized fuzzy metric space $(X, M, *)$ are called compatible if $M(fgx_n, gfx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $fx_n = gx_n$ for some x in X.

Definition 2.7. Two self maps *f* and *g* of a generalized fuzzy metric space $(X, M, *)$ are called reciprocally continuous on X, if $fgx_n = fx$ and $gfx_n = gx$, whenever $\{x_n\}$ is a sequence in X such that $fx_n = gx_n = x$ for some x in X.

Definition 2.8. Let $(X, M, *)$ be a generalized fuzzy metric space. If there exists $q \in (0, 1)$, such that $M(x, y, z, qt) \geq M(x, y, z, t)$ for all $x, y, z \in X$ and $t > 0$.

Definition 2.9. Let X be a set and f, g self maps of X. A point x in X is called coincidence point of *f* and, iff $fx = gx$. We say $w = fx = gx$, a point of coincidence of *f* and g.

Definition 2.10. Two self maps A and B of a generalized fuzzy metric space $(X, M, *)$ are called weak-compatible (or coincidentally commuting) if they commute at their coincidence point, i.e., if $Ax = Bx$ then $ABx = BAx$ for some $x \in X$.

Lemma 2.11. *Let X be a set f, OWC self maps of X. If f and g have a unique point of coincidence,* $w = fx = gx$, *then w is the unique common fixed point of f and g.*

3. Main Results

In this section we prove some fixed point theorems satisfying occasionally weakly compatible mappings common in generalized fuzzy metric spaces. In fact we prove following results.

Theorem 3.1. *Let $(X, M, *)$ be a complete generalized fuzzy metric space and let A, B, C, S and T, U be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}, \{C, U\}$ are OWC. If there exists a point $q \in (0, 1)$, for all $x, y, z \in X$ and $t > 0$, such that*

$$M(Ax, By, Cz, qt) \geq \alpha_1 \min\{M(Sx, Ty, Uz, t), M(Sx, Ax, Sx, t)\}$$

$$+\alpha_2 \min\{M(Ty, By, Ty, t), M(Ax, Ty, Uz, t)\}$$

$$+\alpha_3(M(By, Sx, Uz, t)) + \alpha_4(M(x, By, Cz, t)), \tag{3.1.1}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$, and $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1$. Then there exists a unique point of $w \in X$, such that $Aw = Sw = w$ and a unique point $v \in X$, such that $Bv = Tv = v$. Moreover $v = w$, so that there is a unique common fixed point of A, B, C, S and T, U.

Proof. Let the pairs $\{A, S\}$ and $\{B, T\}, \{C, U\}$ be OWC, there are points $x, y, z \in X$ such that $Ax = Sx$ and $By = Ty$, we claim that $Ax = By, Cz = Uz$. From 3.1.1 we have

$$M(Ax, By, qt) \geq \alpha \min\{M(Sx, Ty, Uz, t), M(Sx, Ax, Sx, t)\}$$

$$+\alpha_2 \min\{M(Ty, By, Ty, t), M(Ax, Ty, Uz, t)\}$$

$$+\alpha_3(M(By, Sx, Uz, t)) + \alpha_4(M(Ax, By, Cz, t))$$

$$= \alpha_1 \min\{M(Ax, By, Cz), M(Ax, Ax, Ax, t)\}$$

$$+\alpha_2 \min\{M(By, By, By, t), M(Ax, By, Cz, t)\}$$

$$= \alpha_1 \min\{M(Ax, By, Cz, t), 1\}$$

$$+\alpha_2 \min\{1, M(Ax, By, Cz, t)\}$$

$$+\alpha_3(M(Ax, By, Cz, t)) + \alpha_4(M(Ax, By, Cz, t))$$

$$= (\alpha_1 + \alpha_2 + \alpha_3 + +\alpha_4)(M(Ax, By, Cz, t)).$$

A contradiction, since $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1$. Therefore $\alpha_1 = \alpha_2 = Cz$, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = Ty = Cz = Uz$. Suppose that there is a another point v

such that $Av = Sv$ then by 3.1.1. we have $Av = Sv = By = Ty = Cz = Uz$, so $Ax = Av$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. Using Lemma 2.11, we get w is the only common fixed point of A and S, i.e., $w = Aw = Sw$. Similarly there is a unique point $v \in X$ such that $v = Bv = Tv$ and there is a unique point $r \in X$ such that $r = Cr = Ur$. Assume that $w \neq v$. We have

$$M(w, v, r, qt) = M(Aw, Bv, Cr, qt)$$

$$\geq \alpha \min \{M(Sw, Bv, Ur, t), M(Sw, Av, Sw, t)\}$$

$$+ \alpha_2 \min\{M(Tv, Bv, Tv, t), M(Aw, Tv, Ur, t)\}$$

$$+ \alpha_3(M(Bv, Sw, Ur, t)) + \alpha_4(M(Aw, Bv, Cr, t))$$

$$= \alpha_1 \min \{M(w, v, r, t), M(w, v, w, t)\}$$

$$= \alpha_3(M(v, w, r, t)) + \alpha_4(M(w, v, r, t))$$

$$= \alpha_1 \min \{M(w, v, r, t), 1\}$$

$$+ \alpha_2 \min \{1, (w, v, r, t)\}$$

$$+ \alpha_3 \min (M(w, v, r, t)) + \alpha_4(M(w, v, r, t))$$

$$= \alpha_1(M(w, v, r, t))$$

$$+ \alpha_2(M(w, v, r, t))$$

$$+ \alpha_3(M(w, v, r, t)) + \alpha_4(M(w, v, r, t))$$

$$= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)M(w, v, r, t).$$

Since $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1$. Therefore, a contradiction, we have $v = w$ also v is a common fixed point of A, B, C, S and T, U. Next, Again, we consider, $By = Br$, we show that, $v = r$.

Suppose, $v \neq r$, we have

$$M(w, v, r, qt) = M(Aw, Bv, Cr, qt)$$

$$\geq \alpha_1 \min \{M(Sw, Tv, Ur, t), M(Sw, Aw, Sw, t)\}$$

$$+\alpha_2 \min\{M(Tv, Br, Tv, t), M(Aw, Tv, Ur, t)\}$$

$$+\alpha_3(M(Bv, Sw, Ur, t)) + \alpha_4(M(Aw, v, Cr, t))$$

$$= \alpha_1 \min\{M(w, v, r, t), M(w, w, w, t)\}$$

$$+\alpha_2 \min\{M(v, r, v, t), M(w\ v, r, t)\}$$

$$+\alpha_3(M(v, w, r, t)) + \alpha_4(M(w, v, r, t))$$

$$= \alpha_1 \min\{M(w, v, r, t), 1\}$$

$$+\alpha_2 \min\{1, M(w, v, r, t)\}$$

$$+\alpha_3(M(v, w, r, t)) + \alpha_4(M(w, v, r, t))$$

$$= \alpha_1(M(w, v, r, t)) + \alpha_2(M(w, v, r, t))$$

$$+\alpha_3(M(w, v, r, t)) + \alpha_4(M(w, v, r, t))$$

$$= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)M(w, v, r, t).$$

Since $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1$. Therefore, a contradiction, we have $v = r$ also v is a common fixed point of A, B, C, S and T, U. The uniqueness of the fixed point holds from 2.1(i).

Corollary 3.2. *Let $(X, M, *)$ be a complete generalized fuzzy metric space and let A, B, C and T, U be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}, \{C, U\}$ are OWC. If there exists a point $q \in (0, 1)$, for all $x, y, z \in X$, such that*

$$M(Ax, By, Cz, qt) \geq \alpha M(Sx, Ty, Uz, t)$$

$$+\beta \min\{M(Sx, Ax, Sx, t), M(Ax, Ty, Uz, t)\}$$

$$+\gamma \min\{M(Ty, By, Ty, t), M(By, Sx, Uz, t)\}$$

$$+\delta M(Ax, By, Cz, t), \tag{3.1.2}$$

where $\alpha, \beta, \gamma > 0$, and $(\alpha + \beta + \gamma + \delta) > 1$ then there exist a unique point $w \in X$, such that $Aw = Sw = w$ and a unique point $v \in X$, such that

$Bv = Tv = v$. Moreover $v = w$, so that it is a unique common fixed point of A, B, C, S and T, U.

Corollary 3.3. *Let $(X, M, *)$ be a complete generalized fuzzy metric space and let A and S be self map of X. Let the A and S are OWC. If there exists a point $q \in (0, 1)$ for all $x, y, z \in X$ and $t > 0$, such that*

$$M(Sx, Sy, Sz, qt) \geq \alpha_1 M(Ax, Ay, Az, t) + \alpha_2 M(Ax, Ay, Az, t)$$

$$+\alpha_3 \{M(Sy, Ay, Sy, t), M(Ax, Sy, Sz, t)\} + \alpha_4 M(Ax, Ay, Sz, t), \tag{3.3.1}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$ and $(\alpha_1, \alpha_2 + \alpha_3 + \alpha_4) > 1$. Then A and S have a unique common fixed point.

Corollary 3.4. *Let $(X, M, *)$ be a complete generalized fuzzy metric space and let A, B, C, S and T, U be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}, \{C, U\}$ are OWC. If there exists a point $q \in (0, 1)$, for all $x, y, z \in X$, such that*

$$M(Ax, By, Cz, qt) \geq \delta(\min (M(Sx, Ty, Uz, t)M(Sx, Ax, Sx, t), M(Ty, By, Ty, t)))$$
$$[M(Ax, Ty, Uz, t) + M(By, Sx, Uz, t) + M(Ax, BNy, Cz, t)].$$

Such that $\delta(t) > t$ for all $0 < t < 1$, and $\delta : [0, 1] \to [0, 1]$, then there exists a unique common fixed point of A, B, C, S and T, U.

Corollary 3.5. *Let $(X, M, *)$ be a complete generalized fuzzy metric space and let A, B, C, and T, U be self-mappings of X. Let the pairs (A, S) and $(B, T), (C, U)$ are OWC. If there exists a point $q \in (0, 1)$, such that*

$$M(Ax, By, Cz, qt) \geq \delta\left(\frac{M(Sx, Ty, Uz, t), M(Sx, Ax, Sx, t), M(Ty, By, Ty, t)}{[M(Ax, Ty, Uz, t) + M(By, Sx, Uz, t) + M(Ax, By, Cz, t)]}\right)$$

for all $x, y, z \in X$ and $\delta : [0, 1]^5 \to [0, 1]$ such that $\delta(t, 1, 1, t, t) > t$ for all $0 < t < 1$, then there exists a unique common fixed point of A, B, C, S and T, U.

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