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SUPER EQUITABLE DOMINATION IN GRAPHS

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ABSTRACT. An equitable dominating set D of $V(G)$ is called a super equitable dominating set of G if every vertex of $V-D$ has a private equitable neighbour in D. This paper initiates the study of super equitable dominating set.

1. INTRODUCTION

Let $G=(V,E)$ be a simple graph. A subset D of $V(G)$ is called an equitable dominating set if for every $v \in V-D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d(u)-d(v)| \leq 1$, where $d(u)$ denotes the degree of vertex u and $d(v)$ denotes the degree of vertex v. The minimum cardinality of such a dominating set is called the equitable domination number of G and is denoted by $\gamma_e(G)$.

The equitable neighbourhood of u denoted by $N_e(u)$ is defined as $N_e(u) = \{v \in V | v \in N(u), |d(u)-d(v)| \leq 1\}$ and $|N_e(u)| = d_e(u)$. The maximum and minimum equitable degree of a point in G are denoted respectively by $\Delta_e(G)$ and $\delta_e(G)$.

For $S \subseteq V(G)$ and $u \in S$, the set $pn_e(u,S) = N_e[u] - N_e[S-\{u\}]$ is called the private equitable neighborhood of u with respect to S. The set of all the external private equitable neighbour of u with respect to S is denoted by $epn_e(u,S)$.

A subset D of $V(G)$ is called a super dominating set if for every vertex $v \in V(G)-D$ there exists an external private neighbour of v with respect to

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$V(G) - D$. The minimum cardinality of a super dominating set in G is called the super domination number of G and is denoted by $\gamma_{sp}(G)$.

For further reference see [1]-[5].

2. PRELIMINARIES

Definition 2.1. *Let G be a simple graph. An equitable dominating set D is called a super equitable dominating set if every vertex in V-D has an private equitable neighbour in D. The minimum cardinality of a super equitable dominating set is called super equitable domination number of G and is denoted by* $\gamma_{spe}(G)$.

Remark 2.1. *The property of super equitable domination is super hereditary.*

Observation 1. *A subset* $D \subseteq V(G)$ *is a super equitable dominating set of G if and only if for every* $v \in V - D$*, there exists* $u \in N_e(v) \cap D$ *such that* $N_e(u) \subseteq D \cup \{v\}$.

2.1. $\gamma_{spe}(G)$ for some standard graphs.

(1) $\gamma_{spe}(K_n) = n - 1$

(2) $\gamma_{spe}(K_{1,n}) = \begin{cases} 1, & \text{if } n = 1 \\ 2, & \text{if } n = 2 \\ n+1, & \text{if } n \geq 3 \end{cases}$

(3) $\gamma_{spe}(K_{m,n}) = \begin{cases} m+n-2, & \text{if } |m-n| \leq 1 \\ m+n, & \text{if } |m-n| \geq 2 \end{cases}$

(4) $\gamma_{spe}(P_n) = \gamma_{sp}(P_n) = \lceil \frac{n}{2} \rceil$, $n \geq 3$

(5) $\gamma_{spe}(C_n) = \gamma_{sp}(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } n \equiv 0, 3 (mod 4) \\ \lceil \frac{n+1}{2} \rceil, & \text{otherwise} \end{cases}$

(6) $\gamma_{spe}(W_n) = \begin{cases} 3, & \text{if } n = 4 \text{ or } 5 \\ \gamma_{sp}(C_{n-1}) + 1, & \text{if } n \geq 6 \end{cases}$

(7) $\gamma_{spe}(D_{r,s}) = \begin{cases} r+s+1, & \text{if } |r-s| \leq 1, r, s \geq 2 \\ r+s+2, & \text{if } |r-s| \geq 2, r, s \geq 2 \\ 2, & \text{if } r = 0, s = 1 \text{ or } r = 1, s = 0 \text{ or } r = 1, s = 1 \end{cases}$

3. RESULTS ON γ_{spe}

Proposition 3.1. *Let D be a super equitable dominating set of G. D is a minimal super equitable dominating set of G if and only if for every $v \in D$ at least one of the following holds.*

(i) $pn_e[v, D] \neq \phi$.

(ii) *there exists a vertex $w \in (V - D) \cup \{v\}$ such that $epn_e(w, (V - D) \cup \{v\}) = \phi$.*

Proof. Suppose D is a minimal super equitable dominating set of G. Let $v \in D$. Then by Remark 2.1, $D - \{v\}$ is not a super equitable dominating set of G. Suppose $D - \{v\}$ is not an equitable dominating set of G, then $pn_e(v, D) \neq \phi$. If $D - \{v\}$ is an equitable dominating set, then it is not a super equitable dominating set. Therefore, there exists some $w \in (V - D) \cup \{v\}$ such that $epn_e(w, (V - D) \cup \{v\}) = \phi$.

Conversely, let D be a super equitable dominating set of G such that one of the two condition holds for any $v \in D$. If condition i) holds then $D - \{v\}$ is not an equitable dominating set. If condition ii) holds, then $(V - D) \cup \{v\}$ contains a vertex which has no private equitable neighbour in $D - \{v\}$. Therefore, D is a minimal super equitable dominating set of G. □

Observation 2. $\gamma_{spe}(G) \geq \left\lceil \frac{n}{2} \right\rceil$.

Proof. Let D be a γ_{spe} - set. Since every vertex of $V - D$, has an private equitable neighbour in D, $|D| \geq |V - D|$. Therefore, $\gamma_{spe}(G) \geq \left\lceil \frac{n}{2} \right\rceil$. □

Observation 3. $\gamma_{spe}(G) = 1$ *if and only if* $G \simeq K_1$ *or* K_2 .

Observation 4. $\gamma_{spe}(G) = n$ *if and only if every vertex of G is an equitable isolate.*

Observation 5. *For any graph G without equitable isolates,* $1 \leq \gamma_e(G) \leq \frac{n}{2} \leq \gamma_{spe}(G) \leq n - 1$.

Remark 3.1. *Any equitable dominating set of cardinality less than $\frac{n}{2}$ is not a super equitable dominating set.*

Observation 6. *There is no relationship between* $\gamma_{sp}(G)$ *and* $\gamma_{spe}(G)$.

Example 1. *In Figure 1,* $\gamma_{sp}(G_1) = 4 = \gamma_{spe}(G_1)$, $\gamma_{sp}(G_2) = 5 > 4 = \gamma_{spe}(G_2)$ *and* $\gamma_{sp}(G_3) = 4 < 5 = \gamma_{spe}(G_3)$.

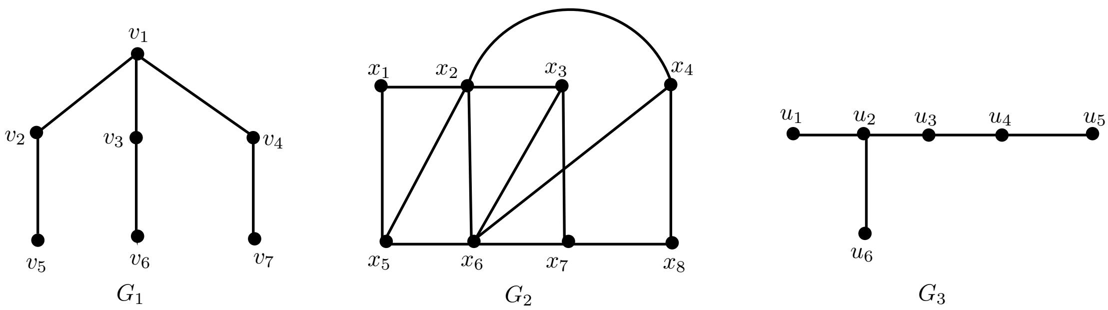

FIGURE 1

Definition 3.1. *A matching M is perfect if every vertex is at an end of an edge in M. A perfect matching M is said to be equitable if the end vertices of every edge in M are degree equitable in G.*

Example 2.

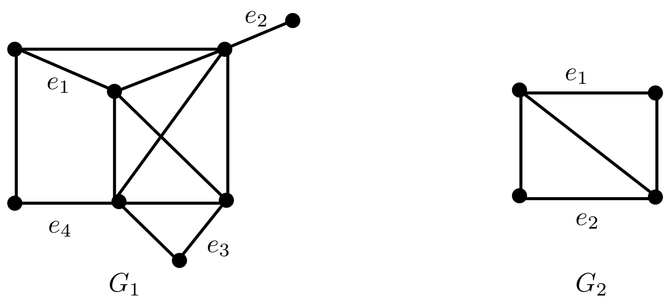

FIGURE 2

In Figure 2, the set of edges $\{e_1, e_2, e_3, e_4\}$ of G_1 is a perfect matching but not equitable. The set of edges $\{e_1, e_2\}$ of G_2 is a perfect matching which is equitable.

Theorem 3.1. *Let G be a simple graph without equitable isolates.* $\gamma_{spe}(G) = \frac{n}{2}$ *if and only if there exists a minimum super equitable dominating set D such that E(D, V-D) is an equitable perfect matching.*

Proof. Suppose there exists a γ_{spe} - set D such that $E(D, V - D)$ is an equitable perfect matching. Then $|D| = |V - D|$, therefore $\gamma_{spe} = \frac{n}{2}$.

Conversely, suppose $\gamma_{spe} = \frac{n}{2}$. Let D be a minimum super equitable dominating set of G. If there exists a vertex $y \in D$ such that $|N_e(y) \cap (V - D)| > 1$, then by definition and since G has no equitable isolates we obtain $|D| > |V - D|$, a contradiction. If there exists a vertex $y \in D$ such that $|N_e(y) \cap (V - D)| = 0$, then either G has an equitable isolate or there is a vertex $z \in D$ such that z has more than one equitable neighbour in $V - D$, a contradiction. Thus every vertex has exactly one neighbour in $V - D$.

Suppose there is a vertex $w \in V - D$ such that $|N_e(w) \cap D| \geq 2$. Then either G contains an equitable isolate or $|D| > \frac{n}{2}$ or there exists a vertex from D which has more than one neighbour in $V - D$, a contradiction. Thus every vertex of $V - D$ has exactly one neighbour in D. Thus, $E(D, V - D)$ is an equitable perfect matching. □

Definition 3.2. *A graph G is said to be equitably connected if any two vertices of G are connected by a path in which every two consecutive vertices are degree equitable in G. The maximum length of an equitable path in an equitably connected graph is called the equitable diameter and is denoted by* $diam_{eq}(G)$.

Remark 3.2. *If G is an equitable graph, then the path connecting any two vertices of G is equitable.*

Lemma 3.1. *Let G be an equitably connected graph with* $diam_{eq}(G) \geq 3$*. Then,* $\gamma_{spe}(G) \leq n - 2$.

Proof. Let $diam_{eq}(G) = k$. Clearly, $k \geq 3$. Let x, y be two vertices of G such that the $d_{eq}(x, y) = k$. Let $\{x, y_1, y_2, \cdots, y_{k-1}, y\}$ be an equitable diametrical path. Then, $V - \{x, y\}$ is a super equitable dominating set of G and hence, $\gamma_{spe} \leq n - 2$. □

Corollary 3.1. *Let G be an equitably connected graph. If* $\gamma_{spe}(G) = n - 1$*, then* $diam_{eq}(G) \leq 2$.

Observation 7. *The converse of Lemma 3.1 is not true.*

Lemma 3.2. *Let G be an equitably connected graph of order* $n \geq 2$*. Then,* $\gamma_{spe}(G) \leq 2m - n + 1$*. If equality holds, then G is an equitably connected tree.*

Proof. $\gamma_{spe}(G) \leq n - 1 = 2(n - 1) - n + 1 \leq 2m - n + 1$. If $\gamma_{spe}(G) = n - 1$, then $m = n - 1$ and hence G is an equitably connected tree. □

Theorem 3.2. *For any graph G,* $\gamma_{spe} \geq n - \frac{1}{2} - \sqrt{\frac{2n^2 - 2n - 4m + 1}{4}}$.

Proof. Let D be a γ_{spe} - set of G. Then, for every $u \in V - D$ there exists $v \in N_e(u) \cap D$ such that $N_e(v) \subseteq D \cup \{u\}$. For every $u \in V - D$, we can find an element v equitably adjacent to u. Hence, v is not equitably adjacent to $n - \gamma_{spe} - 1$ vertices of $V - D$. Since there are $n - \gamma_{spe}$ vertices in $V - D$, we can find $n - \gamma_{spe}$ vertices in D such that each of $n - \gamma_{spe}$ vertices in D is not equitably adjacent to $n - \gamma_{spe} - 1$ vertices in $V - D$. Therefore,

$$m \leq \frac{n(n-1)}{2} - (n - \gamma_{spe})(n - \gamma_{spe} - 1)$$
$$\implies \gamma_{spe} \geq n - \frac{1}{2} - \sqrt{\frac{2n^2 - 2n - 4m + 1}{4}}.$$

□

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