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FEEDBACK RESOLVING SETS IN GRAPHS

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ABSTRACT. For an ordered subset $W = \{w_1, w_2, ..., w_k\}$ of vertices in a connected graph G and a vertex v of G, the *metric representation* of v with respect to W is the k-vector $r(v/W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$. The set W is a *resolving set* for G if $r(u/W) \neq r(v/W)$ for every pair of distinct vertices u and v of G. A resolving set D such that $\langle V - D\rangle$ is acyclic is called a *feedback resolving set*. The minimum and maximum cardinality of a minimal feedback resolving set are denoted by β_* and β_*^+ respectively. This paper initiates a study on these parameters.

1. INTRODUCTION

By a graph $G = (V, E)$, we mean a connected, finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartand and Lesniak [3].

The $distance$ $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u{-}v$ path in G. For an ordered subset $W = \{w_1, w_2, ..., w_k\}$ of vertices in a connected graph G and a vertex v of G, the *metric representation of* v with respect to W is the k-vector $r(v/W) = (d(v, w_1)$,$d(v, w_2)$,..., $d(v, w_k))$. The set W is a *resolving set* for G if $r(u/W) \neq r(v/W)$ for every pair of distinct

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vertices u and v of G. The *metric dimension* $\beta(G)$ of G is the minimum cardinality of a resolving set for G while the upper metric dimension $\beta^+(G)$ is the maximum cardinality of a minimal resolving set in G. The concept of resolving set and the respective parameter metric dimension of a graph was introduced by Slater in the name of locating set and location number respectively. Later, Harary and Melter [7] also independently discovered the same notion, but used the term metric dimension, rather than location number, the terminology that we have adopted. For more details about the notion of metric dimension, one can refer to [1, 2, 4] and [6]. Recently, these concepts have been extended in various ways. For example, connected resolving set [9], independent resolving set [5] and acyclic resolving set [8] are some such concepts. In this sequence, this paper introduces and studies the concept of feedback resolving set.

2. Definitions and examples

In this section, we define the notion of feedback resolving set of a graph and the corresponding parameter, namely feedback metric dimension and upper feedback metric dimension. Further, we determine the feedback metric dimension for some common classes of graphs such as paths, cycles, complete graphs, complete bipartite graphs, the Petersen graph and wheels.

Definition 2.1. *A resolving set S of a graph G such that $\langle V - S\rangle$ is acyclic is said to be a $feedback\ resolving\ set$ (FR-set). A feedback resolving set none of whose proper subsets is a feedback resolving set is a $minimal\ feedback\ resolving\ set$ (MFR-set). The* feedback metric dimension $\beta_*(G)$ *and the* upper feedback metric dimension $\beta_*^+(G)$ *are respectively the minimum and maximum cardinality of a MFR-set of G.*

Remark 2.1.
(i) *In a non-trivial graph G, the set $V(G) - \{v\}$ is always a FR-set, for any vertex v in G. This implies that any non-trivial graph possesses a FR-set and therefore the parameters $\beta_*(G)$ and $\beta_*^+(G)$ are well-defined for any non-trivial graph.*
(ii) *Certainly, a subset $D \subseteq V(G)$ containing a FR-set is also a FR-set so that the property being feedback resolvability is super heriditery. Therefore, a FR-set S of G is minimal if and only if $S - \{v\}$ is not a FR-set of G, for any vertex $v \in S$.*

Example 1. (i) *Since a tree T is acyclic, it follows that $\beta_*(T) = \beta(T)$ and $\beta_*^+(T) = \beta^+(T)$.*

(ii) *Consider the graph G given in Figure 1. Here the set $S = \{x_1, y_2, y_3, x_4, v_2, v_3\}$ is a FR-set of G so that $\beta_*(G) \leq 6$. On the other hand, consider a FR-set D of G. Since $\langle V - D\rangle$ is acyclic, the set D must contain at least two of the vertices v_1, v_2, v_3, v_4 and v_5. Further, for a vertex $v \in V(G)$, we have $d(x_i, v) = d(y_i, v)$, for all $i \in \{1, 2, 3, 4\}$. Therefore, for each $i \in \{1, 2, 3, 4\}$, either x_i or y_i (possibly both) must lie in D. Thus $|D| \geq 6$ and so $\beta_*(G) = 6$. Further, the set $D = \{x_1, x_3, x_4, y_2, y_3, y_4, v_2, v_5\}$ is a MFR-set of maximum cardinality so that $\beta_*^+(G) = 8$.*

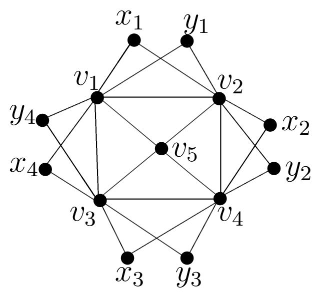

FIGURE 1. A graph G with $\beta_*(G) = 6$ and $\beta_*^+(G) = 8$

In the following, we determine the value of β_* for some common classes of graphs such as cycles, complete graphs, complete bipartite graphs, wheels and the Petersen graph. In a graph G, a vertex v is said to $resolve$ the vertices u and w if $d\,(v, u) \neq d\,(v, w)$.

Proposition 2.1. (i) *The feedback metric dimension of a cycle is always two.*

(ii) *For the complete graph K_n, $(n \geq 2)$, $\beta_*(K_n) = n - 1$.*

Proof. (i) It is clear that the set consisting of any two adjacent vertices of a cycle C_n is a FR-set of the cycle so that $\beta_*(C_n) \leq 2$. Further, a single vertex will not resolve its two neighbours so that a FR-set must contain at least two vertices. Thus $\beta_*(C_n) = 2$.

(ii) For any vertex v of K_n, the set $V(K_n) - \{v\}$ is a FR-set of K_n so that $\beta_*(K_n) \leq n - 1$. Further, since any two vertices of K_n are adjacent, for any resolving set S of K_n, we have $|V - S| \leq 1$. That is, $\beta_*(K_n) \geq n - 1$. □

Proposition 2.2. *Let r and s be integers with $r, s \geq 1$ and $r + s \geq 3$. Then $\beta_*(K_{r,s}) = r + s - 2$.*

Proof. Let (X, Y) be the bipartition of $K_{r,s}$ with $|X| = r$ and $|Y| = s$. Let $x \in X$ and $y \in Y$. Then $S = V(K_{r,s}) - \{x, y\}$ is a FR-set of $K_{r,s}$ so that $\beta_*(K_{r,s}) \leq r + s - 2$. Now, let S be any FR-set of $K_{r,s}$. We need to prove that $|V - S| \leq 2$. If not, consider any three vertices u, v and w in $V - S$. Assume that $u, v \in X$. Then $d(z, u) = d(z, v)$ for all $z \in S$ and so $r(u/S) = r(v/S)$ which implies that S is no longer a FR-set, a contradiction. Thus $|S| \geq r + s - 2$. Hence $\beta_*(K_{r,s}) = r + s - 2$. □

Proposition 2.3. *The feedback metric dimension of the Petersen graph is three.*

Proof. Let the vertices of the Petersen graph G be labeled as in Figure 2. Then $S = \{v_1, v_4, u_2\}$ is a FR-set of G and so $\beta_*(G) \leq 3$. Further, for any subset $D = \{v_i, v_j\}$ of $V(G)$ with cardinality two, there exist at least two vertices say v_r and v_s such that they are not adjacent to both v_i and v_j. Since diameter of the Petersen graph is two, any two non-adjacent vertices are at a distance of two. Hence neither v_i nor v_j resolve the pair of vertices v_r and v_s. Hence D is not a FR-set so that $\beta_*(G) = 3$. □

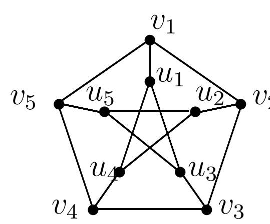

FIGURE 2. The Petersen graph

Proposition 2.4. *If W_n denotes the wheel on n + 1 vertices, then*

$$\beta_*(W_n) = \begin{cases} \beta(W_n) & \text{if } n = 3, 6 \\ \beta(W_n) + 1 & \textit{elsewhere} \end{cases}.$$

Proof. Let v be the central of W_n. When $n = 3$ or $n = 6$ any minimum resolving set of W_n is also a minimum FR-set of W_n and so we have $\beta_*(W_n) = \beta(W_n)$.

Assume $n \notin \{3, 6\}$. It is clear that if S is a minimum resolving set of W_n, then $S \cup \{v\}$ is a FR-set of W_n and so $\beta_*(W_n) \leq \beta(W_n) + 1$. For the other inequality, let us consider a minimum resolving set D of W_n. It is enough to prove that D is not a FR-set of W_n; this will imply that $\beta_*(W_n) > \beta(W_n)$. We first claim that $v \notin D$. On the contrary, suppose $v \in D$. If x and y are any two vertices of W_n outside D, then $d(x, v) = d(y, v)$ and so there exists $z \in D$ such that $d(x, z) \neq d(y, z)$. Thus $r(x/D - \{v\}) \neq r(y/D - \{v\})$; this means that $D - \{v\}$ is also a resolving set of W_n, a contradiction to the minimality of D. Further it has been proved in [1] that $\beta(W_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$ for $n \notin \{3, 6\}$. This implies that there exist two adjacent vertices u and w, different from v, that are not contained in D. Therefore the vertices u, v and w form a triangle in $\langle V - D \rangle$ and so D is not FR-set of W_n. $\square$

3. Bounds and realization theorems

Theorem 3.1. *A FR-set S of a graph G is minimal if and only if for every vertex $u \in S$ one of the following holds.*

- (i) *at least one of the components of $\langle V - S \rangle$ contains at least two neighbours of u in G.*
- (ii) *there exist $v, w \in V - S$ such that $d(v, x) = d(w, x)$, for all $x \in S - \{u\}$.*

Proof. Let S be a FR-set of G. Assume for every $u \in S$, one of (i) and (ii) holds. We have to show that S is minimal. Suppose not. Then for all $u \in S$, $S - \{u\}$ is also a FR-set. This gives that $\langle V - \{S - \{u\}\} \rangle$ is acyclic and $S - \{u\}$ is a resolving set of G. Since $\langle V - \{S - \{u\}\} \rangle$ is acyclic, u can have at most one neighbour in any component of $\langle V - \{S - \{u\}\} \rangle$. This produces a contradiction to (i). Since $S - \{u\}$ is a resolving set of G, for any two vertices $v, w \in V - \{S - \{u\}\}$, we have $d(v, x) \neq d(w, x)$ for some $x \in S - \{u\}$. This gives a contradiction to (ii). Hence S is minimal.

Conversely, suppose there is a vertex $u \in S$ to which neither (i) nor (ii) holds. We claim that $S - \{u\}$ is a FR-set of G. Since (i) does not hold, u has at most one neighbour in each component of $\langle V - S \rangle$ and so $\langle V - (S \cup \{u\}) \rangle$ is acyclic. Further, as (ii) does not hold, for every pair of vertices v and w in $V - S$, there exists a vertex x in $S - \{u\}$ such that $d(v, x) \neq d(w, x)$ so that $S - \{u\}$ is a resolving set of G. Thus $S - \{u\}$ is a FR-set of G. $\square$

Theorem 3.2. *If G is a connected graph on n vertices, then 1 $\leq \beta_*(G) \leq n-1$. Further $\beta_*(G)$ = 1 if and only if G is a path and $\beta_*(G) = n-1$ if and only if G is complete.*

Proof. As for any vertex $v \in G$, the set $V - \{u\}$ is a FR-set of G, it follows that $\beta_*(G) \leq n-1$. Certainly $\beta_*(G) \geq 1$. Now, suppose $\beta_*(G) = 1$. If $\{v\}$ is a FR-set of G, then for every pair of distinct vertices u and w of G, we have $d\,(u, v) \neq d\,(w, v)$ so that the diameter of G is $n-1$ and consequently G is a path. Also, the set consisting of a single pendant vertex of the path forms a FR-set and so the value of β_* for the path is 1.

Now, suppose $\beta_*(G) = n-1$. If G is not complete, then G contains two vertices u and v with $d\,(u, v) = 2$. Consider an u-v path (u, x, v) in G and let $S = V(G) - \{x, v\}$. Since $d\,(u, x) = 1$ and $d\,(u, v) = 2$, we have S is a FR-set of G. This gives $\beta_*(G) \leq n-2$, a contradiction. Hence $G \cong K_n$. Converse follows from Proposition 2.4. □

Theorem 3.3. *Given positive integers a and n with 1 $\leq a \leq n-1$, there exists a graph G of order n such that $\beta_*(G) = a$.*

Proof. Suppose a and n are two integers with $1 \leq a \leq n-1$. We construct a graph G of order n such that $\beta_*(G) = a$ as follows. Let $G = P_n$ when $a = 1$; let $G = C_n$ when $a = 2$; let $G = K_n$ when $a = n-1$ and let $G = K_{r,s}$ with $r + s = n$ when $a = n-2$. Assume that $3 \leq a \leq n-3$.

Case 1. $n - a$ is odd

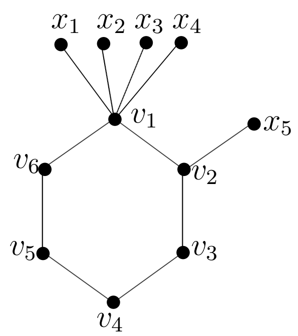

FIGURE 3. A graph G of order 11 with $\beta_*(G) = 5$.

In this case, let G be the graph obtained from the cycle $C_{n-a+1} = (v_1, v_2, ..., v_{n-a+1}, v_1)$ by attaching $a-1$ pendant edges at exactly one of the vertices of the cycle, say v_1. Let us first verify that $\beta_*(G) = a$. If $X = \{x_1, x_2, ..., x_{a-1}\}$ is the set of all pendant vertices of G, then it can be easily verified that $S = \{x_1, x_2, ..., x_{a-2}, v_2, v_{n-a+1}\}$ is a FR-set of G and so $\beta_*(G) \leq a$. For the other inequality, consider FR-set S of G. Then S contains at least one vertex of the cycle C_{n-a+1} as $d(v_2, x_i) = d(v_{n-a+1}, x_i)$, for all $i \in \{1, 2, .., a-1\}$. Further, if there exist two vertices x_i and x_j that are not contained in S, then $d(x_i, v) = d(x_j, v)$, for all $v \in S$ and so $r(x_i/S) = r(x_j/S)$, a contradiction to the fact that S is FR-set of G. Therefore S contains at least $a-2$ vertices from the set X. Now, if S contains exactly $a-1$ vertices from X, then $\beta_*(G) \geq a$ as desired. Suppose S contains $a-2$ vertices from X. In this case we show that S contains at least two vertices of the cycle C_{n-a+1}. If not, then S contains exactly one vertex of the cycle C_{n-a+1}. Let the vertex be x. Suppose x is either v_1 or $v_{\frac{n-a+1}{2}}$. Then $d(v_{n-a+1}, u) = d(v_2, u)$, for every $u \in S$ so that $r(v_{n-a+1}/S) = r(v_2/S)$ which implies that S is no longer a FR-set of G, a contradiction. If $x \in \{v_2, v_3, ..., v_{\frac{n-a+1}{2}-1}\}$, then $d(x_{a-1}, w) = d(v_{n-a+1}, w)$, for all $w \in S$, again a contradiction. If $x \in \{v_{\frac{n-a+1}{2}+1}, v_{\frac{n-a+1}{2}+2}, ..., v_{n-a+1}\}$, then $d(x_{a-1}, y) = d(v_2, y)$, for every $y \in S$, a contradiction. Therefore S contains at least two vertices lying on the cycle C_{n-a+1} and so $\beta_*(G) \geq a$.

Case 2. $n-a$ is even

Here, let G be the graph obtained from the cycle $C_{n-a} = (v_1, v_2, ..., v_{n-a}, v_1)$ by attaching $a-1$ pendant edges at any one of the vertices of the cycle, say v_1 and attach one pendant edge at any one of the remaining vertices of the cycle say v_2. Let x_1, x_2,..., x_{a-1}, x_a be the pendant vertices of G, where x_a is incident with the pendant edge which is attached at v_2. For $n = 11$ and $a = 5$, the graph G is given in Figure 3. Here, $\beta_*(G) = a$. Clearly $S = \{x_1, x_2, ..., x_{a-2}, x_a, v_{n-a}\}$ is a FR-set of G and so $\beta_*(G) \leq |S| = a$. In a similar argument, the other inequality that $\beta_*(G) \geq a$ can also be proved and thus $\beta_*(G) = a$. □

It follows immediately from the definition that $\beta(G) \leq \beta_*(G)$ for any graph G. Further, if $\beta(G) = 1$, then it has been proved in [2] that G must be a path. However, as in Theorem 3.2, the value of β_* for a path is also 1. But when

$\beta(G) > 1$, then $\beta_*(G)$ can assume any arbitrarily value as shown in the following theorem.

Theorem 3.4. *Given positive integers a and b with $2 \leq a \leq b$, there exists a graph G for which $\beta(G) = a$ and $\beta_*(G) = b$.*

Proof. If $a = b$, let $G = K_{a+1}$. Assume that $a < b$. The required graph G is constructed as follows. Consider a path $P = (v_0, v_1, ..., v_{2b-2a+2})$ on $2b - 2a + 3$ vertices. Attach a complete graph K_{a+1} with the vertex set $\{a_1, a_2, ..., a_{a+1}\}$ at the vertex v_0 so that $a_{a+1} = v_0$. Introduce $b - a + 1$ new vertices, say $u_1, u_2, ..., u_{b-a+1}$ and for each i = 1, 2, ..., $b - a$ + 1, join the vertex u_i to both v_{2i-1} and v_{2i}. Let G be the resultant graph. For a = 4 and b = 7, the graph G is given in Figure 4.

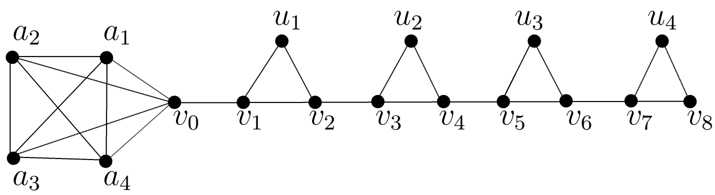

FIGURE 4. A graph G with $\beta(G) = 4$ and $\beta_*(G) = 7$

We first claim that $\beta(G) = a$. Clearly $\{a_1, a_2, ..., a_{a-1}, u_{b-a+1}\}$ is a resolving set of G and so $\beta(G) \leq a$. For the other inequality, consider a resolving set S of G. Then S contains at least $a - 1$ vertices from the set $V(K_{a+1}) - \{a_1\}$ as for any two vertices a_i and a_j in $V(K_{a-1}) - \{a_1\}$, $d(a_i, x) = d(a_j, x)$, for all $x \in S$. Also S contains at least one vertex of the vertices $v_{2a-2a+2}$ and u_{b-a+1}; for otherwise $d(v_{2a-2a+2}, y) = d(u_{b-a+1}, y)$, for every $y \in S$. Hence $\beta(G) \geq a$. We now prove that $\beta_*(G) = b$. Since $\{a_1$, a_2, ..., a_{a-1}, u_1, u_2, ..., $u_{b-a+1}\}$ is a FR-set of G, we have $\beta_*(G) \leq b$. Now, consider a FR-set D of G. Then D contains at least $a - 1$ vertices from the set $V(K_{a+1}) - \{a_1\}$ as D is also a resolving set of G. Further, for each i, $i = 1, 2, ..., b - a + 1$, D contains at least one of the vertices u_i, v_{2i-1} and v_{2i}; for otherwise $\langle V - D\rangle$ contains a triangle. Hence $\beta_*(G) \geq b$. $\square$

Remark 3.1. *It follows from the definitions that $\beta(G) \leq \beta_*(G) \leq \beta_*^+(G)$. These inequalities are strict in the sense that they all may be equal or they all can be distinct. For example, for the graph G of Figure 1, we have $\beta(G) = 4$, $\beta_*(G) = 6$ and $\beta_*^+(G) = 8$. On the other hand, for trees all the three parameters are equal.*

Further, the parameter $\beta^+(G)$ is always greater than or equal to $\beta(G)$; whereas *it has no relation with $\beta_*(G)$. That is, for any graph G, either $\beta^+(G) \leq \beta_*(G)$ or $\beta_*(G) \leq \beta^+(G)$. The graph G of Figure 1 satisfies the first inequality. For a path P_n of order n, $\beta_*(P_n) = 1$ and $\beta^+(P_n) = 2$. However, it seems that $\beta^+(G)$ is always less than or equal to $\beta_*^+(G)$ and we pose this as a conjecture.*

Conjecture 3.1. *For any connected graph G, we have $\beta^+(G) \leq \beta_*^+(G)$.*

4. Conclusion

We conclude this paper by listing some open problems that are encountered during this course of study.

1. Characterize the graphs G for which
   (i) $\beta(G) = \beta_*(G)$.
   (ii) $\beta_*(G) = \beta_*^+(G)$.
2. Given positive integers a, b and c, does there exist a graph G with $\beta(G) = a$, $\beta_*(G) = b$ and $\beta_*^+(G) = c$?
3. Given positive integers r and s, do there exist graphs G_1 and G_2 such that
   (i) $\beta_*(G_1) - \beta^+(G_1) = r$.
   (ii) $\beta^+(G_2) - \beta_*(G_2) = s$.

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